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The Symmetric Vibrations of Aircraft

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(*Summary*: The eigenvalue equations for symmetric vibration of a complete aircraft are derived in a very general form. The "lumped mass" approximation to the continuous mass distribution is used and sub-matrices are associated with properties of relatively simple branches of the system. The final eigenvalue equations are expressed in terms of these sub-matrices, so that in a numerical application the physical system, as such, is considered only in relation to the properties of the simple branches. It is assumed initially that the aircraft wing and tail have flexural axes of the conventional type, but it is shown in the Appendix that under certain conditions the treatment can be generalised to cover swept and cranked wing aircraft.)

1. Introduction

The calculation of natural frequencies and modes of vibration of a complete aircraft is a complicated and laborious process. It is complicated because of the nature of the dynamical system, and laborious because any representation of the system must involve a relatively large number of degrees of freedom. The labour of computation required in the determination of frequencies and modes of a particular aircraft must be accepted as unavoidable since, whatever the method of solution, the amount of computation is considerable. The derivation of the vibration equations is a task which need be performed but once in general terms to be applicable to the whole field of conventional aircraft. It is the purpose of this paper to develop the eigenvalue equations for symmetric vibrations of the widely used "lumped mass" approximate representation of the continuous system, in a form which is convenient for direct substitution of numerical values of fundamental physical quantities of the system.

The eigenvalue equations are stated in matrix notation[1] and the derivation of these equations is set out in matrix form. This leads to concise statement throughout and is an aid to clarity. Considerable use is made of sub-matrices which are related to specific parts or branches of the system, and the equations finally derived are defined in terms of these matrices. This implies that in an application to a specific aircraft reference is made to the physical system only in the description

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of properties of relatively simple branches and that a purely formal process of matrix multiplication and addition then yields the required equations. Further, the matrices involved in general refer to quite arbitrary numbers of degrees of freedom in the various branches of the system so that it is the form only and not the size (in terms of degrees of freedom) of the system which is specified. Initially it is assumed that wings and tail have flexural axes, but it is shown in the Appendix that this is not essential to the treatment.

The present treatment of the problem is in some ways similar to that proposed by J. Morris and G. S. Green[2]. The same concept of "branch system" is used but whereas Morris and Green determine the frequencies and modes of the branch system under prescribed conditions of constraint (*e.g.* wing, encastré at the root) as a preliminary step in establishing the frequency equation for the complete system, this treatment proceeds directly to a formulation of the eigenvalue equations for the complete system. The main point of variance in the two methods lies in the manner in which the equilibrium conditions are utilised. Morris and Green use the equilibrium conditions to establish the frequency equation when the vibration modes and frequencies of the branch systems are known, while here these conditions lead to the establishment of a generalised inertia matrix. It is not practicable to devote the time and labour to a numerical comparison of the two methods. It is felt, however, that solution by matrix iteration[3] of the equation presented is probably more economical in labour than the method proposed by Morris and Green. This is especially so in the calculation of a small number (say three) of lower order modes and frequencies, only for which, as pointed out later, can calculation be accurate. If, on the other hand, it were practicable to calculate a large number of modes and frequencies the relative expediency of the two methods would be more doubtful.

The "lumped mass" approximation to the mass distribution associated with a mass-less continuous elastic system, which is adopted here and by Morris and Green, is a representation which is commonly used for this type of problem, and one which yields satisfactory comparison with experimental results for modes of lower order when the physical properties of the system are determined with sufficient accuracy. It is well known that agreement between calculated and experimentally determined natural frequencies and associated modes becomes less satisfactory with increase in order of the mode. As pointed out by A. R. Collar[4], one source of error lies in the common practice of neglecting the contribution of shear distortion of beam webs when flexibilities of the structure are calculated; but this source can be eliminated in the calculation of influence coefficients. Further, if influence coefficients are measured directly from the aircraft this question is automatically eliminated—and in fact this source of error is covered by the provision that "the physical property of the system be determined with sufficient accuracy."

Another more serious source of error lies in the phenomenon of shear lag, which source[5, 6] it is virtually impossible to eliminate. Shear lag introduces changes in the stiffness of the system for each mode, the effective stiffness depending on the

shape of the mode in question; so that even if the stiffness is determined directly by static test the effective stiffness in higher order modes is not correctly represented. Although calculated higher order natural frequencies and associated modes have little relation to those of the actual system, this is not to say that valuable data cannot be obtained by calculation and that recourse must be made to some form of experiment in all cases. For dynamical problems in which natural modes of vibration with associated generalised co-ordinates may be used to advantage, the displacement of the system can often be expressed adequately in terms of a small number of lower order modes.

2. Representation of Aircraft

It is assumed that the continuous mass distribution of the aircraft is replaced by a finite number of discrete masses at selected points on the elastic structure. The magnitudes and positions of these masses are selected to give as close an approximation as possible to the continuous distribution. A knowledge of the types of displacement which will occur in the various parts of the system during vibration indicates the types of inertia forces which are of importance, and consequently to some extent prescribes the distribution. In symmetric vibration the fuselage displacements are translational with a small component of rotation about axes parallel to the wing flexural axis. The inertia effect in rotation is known to be small, so that the fuselage mass distribution may be represented by point masses on the centre-line. On the wings and tail the effect of rotary inertia is similarly small so that it is satisfactory to represent the spanwise distribution by masses with zero moment of inertia about chordwise axes. The wings and tail, however, do rotate in the torsional sense, and since the mass is distributed over a finite chord, inertia forces in the form of torques result from acceleration in rotation. It is necessary therefore to adjust the chordwise distribution at spanwise mass stations to take account of the torsional inertia forces, and this can be done in either of two ways. At each spanwise station two point masses may be located on the chord so that their centroid lies at the centroid of the chordwise distribution and their moment of inertia corresponds to that of the chordwise distribution. Alternatively, an appropriate moment of inertia may be assigned to one mass situated at each chordwise centroid. The latter alternative is adopted here as this not only effects an algebraic simplification in the case where a straight flexural axis is assumed to exist, but also provides a convenient set of co-ordinates for a consideration of aerodynamic forces when natural modes of vibration are used in aeroelastic applications.

The elastic properties of the continuous system are retained unaltered, but since the inertia properties are reduced to a finite number of discrete values it is no longer necessary to express the former in continuous form. Influence coefficients, defining the displacements in various mass co-ordinates due to forces appropriate to a particular co-ordinate under prescribed conditions of constraint of the system, then serve to specify the elastic properties.

Figure 1 depicts a typical idealisation of a four-engined aircraft for vibration analysis. It is assumed initially that the wings and tail have straight flexural axes. The fuselage is replaced by a straight mass-less beam with point masses attached, while the wings and tail are represented by straight mass-less beams of known flexural and torsional properties. The wing and tail masses are offset from the elastic beams (which are located at the flexural axes) and with each mass is associated a polar moment of inertia. A further polar moment of inertia is associated with the mass at the tail centre-section to take account of the inertia moment arising from the tail fin in single fin aircraft.

3. Notation and Co-ordinates

Since the system is free-free, *i.e.* is reactionless and has no fixity in space, it is necessary to refer elastic displacements to some arbitrary datum within the system. It is convenient, as will become evident, to select this elastic datum at the wing-fuselage junction station and to measure elastic displacements relative to the tangent plane at this point. Elastically the system is then divided into three; and the fuselage forward of the reference station, the wing, and the part of the system aft of the reference station are to some extent treated separately. This requires differentiation in notation between fuselage properties fore and aft of this point. The subscript "f" refers to quantities relating to the fuselage and the qualifying subscripts "w" and "t" indicate the location relative to the reference station; for example, the subscript "ft" refers to fuselage properties on the tail side of the reference station, while "fw" refers to those forward of the reference station. Properties of the wing and tail are associated with the unqualified subscripts "w" and "t" respectively.

The eigenvalue equations are derived for symmetric vibrations only so that it is not necessary to consider the complete system but merely one half which lies to one side of the plane of symmetry. The plane of symmetry is a vertical plane on the fuselage centre-line, and this implies that fuselage properties be halved throughout. It will be understood therefore that inertia and stiffness properties associated with the fuselage refer to half the corresponding values for the complete aircraft.

Section 3.1.

As indicated by the choice of subscripts, for the definition of properties the system can conveniently be considered in four branches. These are:

(*i*) the wing,
(*ii*) the fuselage forward of the reference station,
(*iii*) the fuselage aft of the reference station,
and (*iv*) the tail,

and the following definitions are in accordance with this branch notion:

$(m_i)_w$ mass at i^{th} wing station
$(p_i)_w$ moment of inertia at i^{th} wing station

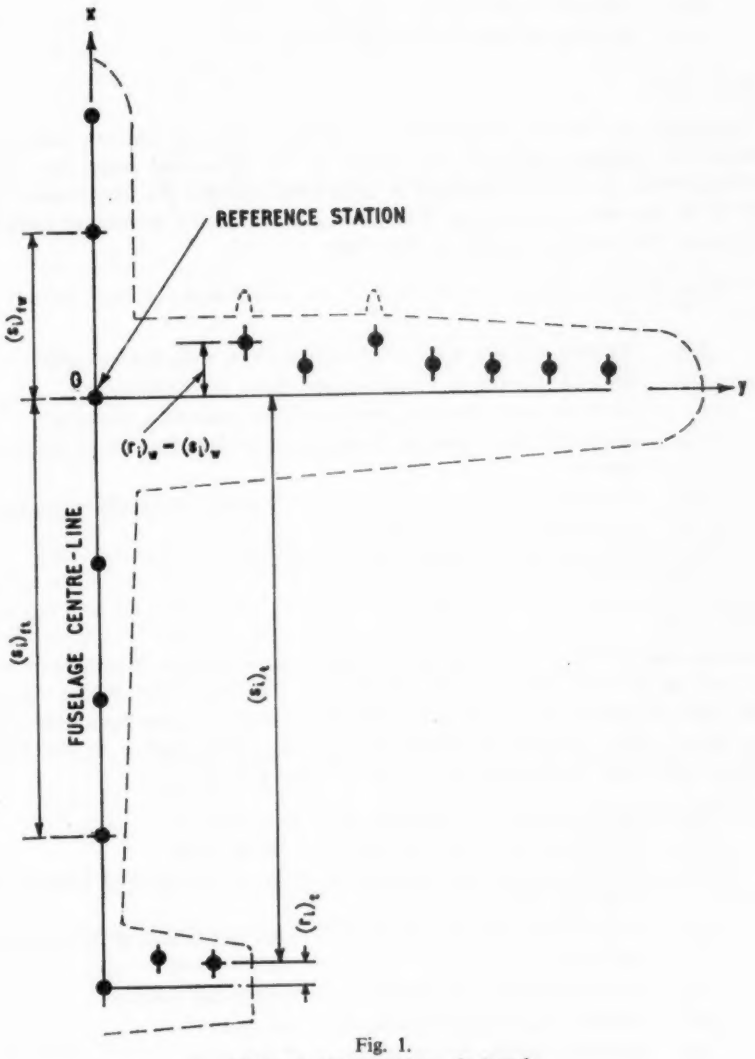

Fig. 1.
Idealised representation of aircraft.

$(m_i)_{tw}$ mass at i^{th} station of fuselage forward of reference station
m_0 mass at reference station
$(m_i)_{ft}$ mass at i^{th} station of fuselage aft of reference station
p_{ft} moment of inertia at fuselage-tail junction

$(m_i)_t$ mass at i^{th} tail station

$(p_i)_t$ moment of inertia at i^{th} tail station.

Section 3.2.

The system is referred to Cartesian co-ordinates, the x co-ordinate measured parallel to the fuselage centre-line and positive in the tail-to-nose sense, the z-axis downwards, and the y-axis completes a right-hand system. In the undistorted condition the aircraft representation is assumed coplanar and is positioned to lie in the xy-plane with reference station at the origin.

Further constants relating to position in the x direction are then defined as follows:

$(r_i)_w$ distance of mass at i^{th} wing station from wing flexural axis

$(r_i)_t$ distance of mass at i^{th} tail station from tail flexural axis (in both cases measured positive in the positive x direction)

$(s_i)_{fw}$ x co-ordinate of mass at i^{th} station of fuselage forward of reference station

$(s_i)_{ft}$ x co-ordinate of mass at i^{th} station of fuselage aft of reference station

$(s_i)_w$ x co-ordinate of mass at i^{th} wing station

$(s_i)_t$ x co-ordinate of mass at i^{th} tail station.

Section 3.3.

Displacements of the system initially appear in the equations of motion in two forms—one as absolute displacements which are associated with inertia forces, and the other as relative displacements within the system which are associated with elastic forces. The absolute displacements are measured relative to the fixed co-ordinate axes and the notation is as follows* (*see* Fig. 2):

$(h_i)_w$ absolute linear displacement of i^{th} wing mass

$(\varphi_i)_w$ absolute angular displacement of i^{th} wing mass

$(h_i)_{fw}$ absolute linear displacement of i^{th} mass of fuselage forward of reference station

$(h_i)_{ft}$ absolute linear displacement of i^{th} mass of fuselage aft of reference station

φ_{ft} absolute angular displacement at fuselage-tail junction

$(h_i)_t$ absolute linear displacement of i^{th} tail mass

$(\varphi_i)_t$ absolute angular displacement of i^{th} tail mass

h_0 absolute linear displacement at reference station

φ_0 absolute angular displacement at reference station.

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*In accordance with the right-hand system of axes, linear displacements are measured positive downwards and angular displacements positive in the nose-up sense.

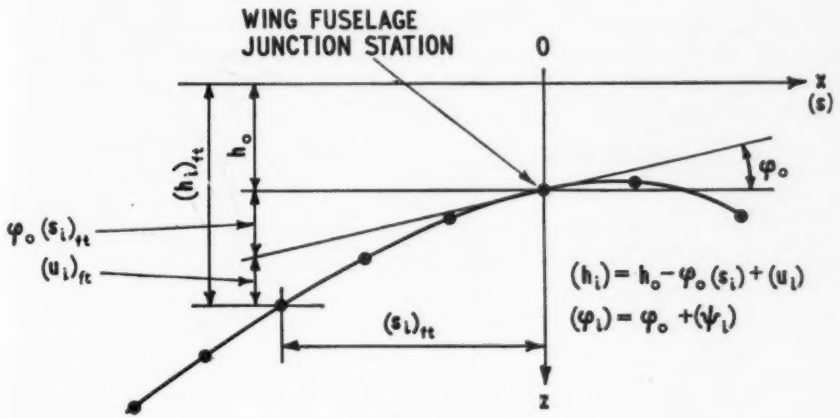

Fig. 2.
Displacement co-ordinates.

Section 3.4.

The displacements relevant to elastic forces are measured relative to the tangent plane at the reference station as shown in Fig. 2. These relative displacements are termed "elastic displacements" and in the same order as the absolute displacements, the co-ordinates are defined as

$(u_i)_w$ elastic linear displacement of i^{th} wing mass

$(\psi_i)_w$ elastic angular displacement of i^{th} wing mass

$(u_i)_{fw}$ elastic linear displacement of i^{th} mass on fuselage forward of reference station

$(u_i)_{ft}$ elastic linear displacement of i^{th} mass on fuselage aft of reference station

ψ_{ft} elastic angular displacement at fuselage-tail junction

$(u_i)_t$ elastic linear displacement of i^{th} tail mass

$(\psi_i)_t$ elastic angular displacement of i^{th} tail mass.

Section 3.5.

The quantities so far defined refer to individual masses of the system and these are now grouped in matrix form to give a more concise notation. Where more than one matrix is associated with a particular branch of the system it is understood that the elements referring to specific stations occur in the same sequence in each matrix. The relevant matrices are defined in Table I.

The orders of these matrices depend on the representation of the system. In addition to those of the table the single term matrices p_{ft}, φ_{ft} and ψ_{ft} are used and these are defined in Sections 3.1, 3.3 and 3.4 respectively. Further, certain "operational" matrices are required and these are defined as

I the unit matrix,

L a column matrix, each element of which is unity,

H a square matrix, each element of which is unity.

The orders of these matrices are nowhere specifically defined, and each may appear as matrices of different order in a single expression. This does not lead to confusion since they occur in association with matrices of known (although here, arbitrary) order, and hence the appropriate order of a particular operational matrix is implied by conditions of compatibility in the operations concerned.

Section 3.6.

It remains to define the elastic properties of the system. In the definitions which follow, square flexibility matrices are matrices of which the element in the i^{th} row and j^{th} column represents the displacement in the i^{th} co-ordinate produced by a unit force in the j^{th} co-ordinate under specified constraint conditions of the system. In this regard "displacement" and "force" may apply either to linear or angular

TABLE I

Diagonal Matrix	*Elements*	*Column Matrix*	*Elements*
A_w	$(m_i)_w$	h_w	$(h_i)_w$
P_w	$(p_i)_w$	φ_w	$(\varphi_i)_w$
A_{tw}	$(m_i)_{tw}$	h_{tw}	$(h_i)_{tw}$
A_{ft}	$(m_i)_{ft}$	h_{ft}	$(h_i)_{ft}$
A_t	$(m_i)_t$	h_t	$(h_i)_t$
P_t	$(p_i)_t$	φ_t	$(\varphi_i)_t$
R_w	$(r_i)_w$	u_w	$(u_i)_w$
R_t	$(r_i)_t$	ψ_w	$(\psi_i)_w$
S_{tw}	$(s_i)_{tw}$	u_{tw}	$(u_i)_{tw}$
S_{ft}	$(s_i)_{ft}$	u_{ft}	$(u_i)_{ft}$
S_w	$(s_i)_w$	u_t	$(u_i)_t$
S_t	$(s_i)_t$	ψ_t	$(\psi_i)_t$

displacement and either linear force or moment respectively. The conditions of constraint are: for the wing and tail, encastré conditions at the root; for the fuselage fore and aft of the reference station, encastré conditions at the reference station.

The matrices are defined as follows:

E_{fw}^{-1}	flexibility matrix of fuselage forward of reference station
E_{ft}^{-1}	flexibility matrix of fuselage aft of reference station
E_w^{-1}	linear displacement flexibility matrix on wing flexural axis
E_t^{-1}	linear displacement flexibility matrix on tail flexural axis
Θ_w^{-1}	angular displacement flexibility matrix of wing
Θ_t^{-1}	angular displacement flexibility matrix of tail
d	column of linear displacements on fuselage aft of reference station due to unit moment at fuselage-tail junction
d'	(transposed of d) row of angular displacements at fuselage-tail junction due to unit linear forces at fuselage masses
e	column of linear displacements on fuselage aft of reference station due to unit linear force at fuselage-tail junction (a column of E_{ft}^{-1})
e'	(transposed of e) row of linear displacements at fuselage-tail junction due to unit linear forces at fuselage masses (a row of E_{ft}^{-1})
δ	linear displacement at fuselage-tail junction due to a unit linear force at this station (an element of E_{ft}^{-1})
β	linear displacement at fuselage-tail junction due to a unit moment at this station $\equiv$ angular displacement at fuselage-tail junction due to a unit linear force at this station
α	angular displacement at fuselage-tail junction due to a unit moment at this station.

4. Inertia Forces

In the free-free vibration of a conservative system the only forces present are elastic and inertia forces. Elastic forces depend solely on the deformation of the system, while inertia forces are proportional to absolute accelerations. D'Alembert's principle makes it possible to write down immediately the inertia force acting at any mass as

$$\text{force} = -(\text{mass}) \times (\text{acceleration}) \qquad \text{for linear motion,}$$

and $$\text{moment} = -(\text{moment of inertia}) \times (\text{angular acceleration}) \qquad \text{for angular motion.}$$

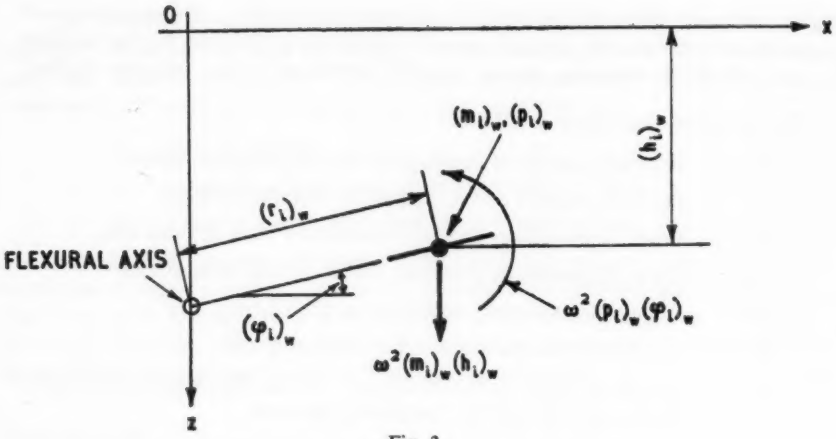

Fig. 3.
Inertia forces at the i^{th} wing station.

In the case of simple harmonic motion at a frequency of ω radians per second these become respectively:

$$\text{force} = \omega^2 \times (\text{mass}) \times (\text{displacement})$$

and $$\text{moment} = \omega^2 \times (\text{moment of inertia}) \times (\text{angular displacement}).$$

The elastic displacements in the various branches of the system resulting from inertia forces in vibration are now deduced in terms of these inertia forces.

As a preliminary to the deduction of elastic displacements it is of advantage to consider in detail the inertia forces on the wing and tail. Fig. 3 depicts the conditions at a spanwise station of the wing. The linear inertia force at the i^{th} station is:

$$(F_i)_w = \omega^2 (m_i)_w (h_i)_w , \quad . \quad . \quad . \quad (1)$$

and the moment about the flexural axis is

$$(M_i)_w = \omega^2 \{ (p_i)_w (\varphi_i)_w - (r_i)_w (m_i)_w (h_i)_w \} \quad . \quad . \quad . \quad (2)$$

where F is measured positive in the positive z direction (downwards) and M is measured positive anti-clockwise.

Similar expressions hold for forces at the i^{th} station of the tail. Further, at the tail root (fuselage-tail junction station) the linear force and moment from the tail

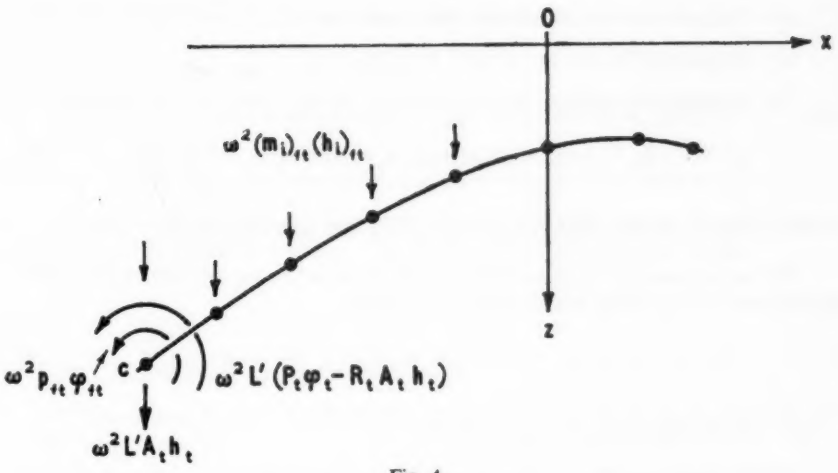

Fig. 4.
Inertia forces on fuselage aft of reference station.

are derived by summation with respect to i. The force and moment from the tail at this point are

$$F = \omega^2 \sum_i (m_i)_t (h_i)_t \quad . \quad . \quad . \quad (3)$$

and $$M = \omega^2 \sum_i \{ (p_i)_t (\varphi_i)_t - (r_i)_t (m_i)_t (h_i)_t \} \quad . \quad . \quad . \quad (4)$$

or $$F = \omega^2 L' A_t h_t \quad . \quad . \quad . \quad (5)$$

and $$M = \omega^2 L' (P_t \varphi_t - R_t A_t h_t) \quad . \quad . \quad . \quad (6)$$

from Table I, where L' is the transposed of L.

5. Elastic Displacements

The elastic displacements in various co-ordinates are as follows, where for convenience the fuselage-tail junction station is referred to as the point "C."

(*i*) *Elastic linear displacements of fuselage aft of reference station*

Figure 4 illustrates the forces applied to this part of the fuselage. The elastic displacements consist of

(*a*) displacements due to forces from fuselage masses,

(*b*) displacements due to moment from moment of inertia at C,

(*c*) displacements due to linear force from tail at *C*,

(*d*) displacements due to moment from tail at *C*.

The displacement matrix with components in the order (*a*)–(*d*) becomes

$$u_{ft} = \omega^2 \{ E_{ft}^{-1} A_{ft} h_{ft} + d p_{ft} \varphi_{ft} + e L' A_t h_t + d L' (P_t \varphi_t - R_t A_t h_t) \} \quad . \quad (7)$$

(*ii*) *Elastic angular displacement of fuselage-tail junction station*

The components of displacement are described as under (*a*)–(*d*) of (*i*). With components in this order the displacement becomes

$$\psi_{ft} = \omega^2 \{ d' A_{ft} h_{ft} + \alpha p_{ft} \varphi_{ft} + \beta L' A_t h_t + \alpha L' (P_t \varphi_t - R_t A_t h_t) \} \quad . \quad (8)$$

(*iii*) *Elastic linear displacements of tail*

Displacements on the tail depend not only on displacements relative to the tail root but also on the linear and angular displacements at the root. The components of tail linear displacements are as follows

(*a*) tail flexural axis displacements at spanwise mass stations relative to tail root,

(*b*) displacements relative to root due to torsion about flexural axis,

(*c*) displacement at *C* (this is included in (*i*) but requires to be specified separately. It consists of components as under (*a*)–(*d*) of (*i*)),

(*d*) displacements due to angular displacement at *C* (the angular displacement at *C* is ψ_{ft} of (*ii*)).

With components in the order (*a*)–(*d*) the displacement matrix then is

$$\begin{aligned} u_t = \omega^2 [& E_t^{-1} A_t h_t - R_t \Theta_t^{-1} (P_t \varphi_t - R_t A_t h_t) + \\ & + L \{ e' A_{ft} h_{ft} + \beta p_{ft} \varphi_{ft} + \delta L' A_t h_t + \beta L' (P_t \varphi_t - R_t A_t h_t) \} - \\ & - R_t L \{ d' A_{ft} h_{ft} + \alpha p_{ft} \varphi_{ft} + \beta L' A_t h_t + \alpha L' (P_t \varphi_t - R_t A_t h_t) \}] \end{aligned} \quad (9)$$

(*iv*) *Elastic angular displacements on tail*

The angular displacements on the tail similarly depend on the angular displacement at the root. The two components of displacement are

(*a*) angular displacement relative to root,

(*b*) angular displacement at *C* (ψ_{ft} of (*ii*)),

and with components in this order the displacements are

$$\psi_t = \omega^2 [\Theta_t^{-1} (P_t \varphi_t - R_t A_t h_t) + \\ + L \{ d' A_{ft} h_{ft} + \alpha p_{ft} \varphi_{ft} + \beta L' A_t h_t + \alpha L' (P_t \varphi_t - R_t A_t h_t) \}] \quad (10)$$

(*v*) *Elastic linear displacements of fuselage forward of reference station*

These displacements are due solely to linear inertia forces on the fuselage forward of the reference station. The elastic displacement matrix is then

$$u_{fw} = \omega^2 E_{fw}^{-1} A_{fw} h_{fw} \quad . \quad . \quad . \quad (11)$$

(*vi*) *Elastic linear displacements on wing*

These are of the same form as the displacements relative to the root on the tail. The components are

(*a*) wing flexural axis displacements

(*b*) displacements due to torsion about flexural axis.

The linear displacement matrix is

$$u_w = \omega^2 \{ E_w^{-1} A_w h_w - R_w \Theta_w^{-1} (P_w \varphi_w - R_w A_w h_w) \} \quad . \quad . \quad (12)$$

(*viii*) *Elastic angular displacements on wing*

Wing elastic angular displacements are measured relative to the root and are given by

$$\psi_w = \omega^2 \Theta_w^{-1} (P_w \varphi_w - R_w A_w h_w) \quad . \quad . \quad . \quad . \quad (13)$$

The equations (7)–(13) define the elastic displacements of the system relative to the tangent plane at the reference station. More concisely they can be expressed in one matrix equation, and it is easily verified that they are equivalent to (14), in which broken lines indicate partitioning into sub-matrices.

6. Equilibrium Conditions and Transformation of Co-ordinates

The equation (14) contains two sets of displacement co-ordinates, the elastic displacement co-ordinates on the left hand side and the absolute displacement co-ordinates on the right hand side. These sets of co-ordinates are related through the equilibrium conditions of the system, *i.e.* the conditions of zero body displacements in translation and rotation; a consideration of these conditions yields the

$$
\begin{bmatrix} u_{\mathrm{ft}} \\ u_{\mathrm{t}} \\ u_{\mathrm{fw}} \\ u_{\mathrm{w}} \\ \psi_{\mathrm{ft}} \\ \psi_{\mathrm{t}} \\ \psi_{\mathrm{w}} \end{bmatrix} = \omega^2 \left[\begin{array}{c|c|c|c|c|c|c} E_{\mathrm{ft}}^{-1} & eL' - dL'R_{\mathrm{t}} & 0 & 0 & d & dL' & 0 \\ \hline Le' - R_{\mathrm{t}}Ld' & \begin{array}{l} E_{\mathrm{t}}^{-1} + R_{\mathrm{t}}\Theta_{\mathrm{t}}^{-1}R_{\mathrm{t}} + \\ +\delta H + \alpha R_{\mathrm{t}}HR_{\mathrm{t}} - \\ -\beta(HR_{\mathrm{t}} + R_{\mathrm{t}}H) \end{array} & 0 & 0 & \beta L - \alpha R_{\mathrm{t}}L & \begin{array}{l} -R_{\mathrm{t}}\Theta_{\mathrm{t}}^{-1} + \\ +\beta H - \alpha R_{\mathrm{t}}H \end{array} & 0 \\ \hline 0 & 0 & E_{\mathrm{fw}}^{-1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E_{\mathrm{w}}^{-1} + R_{\mathrm{w}}\Theta_{\mathrm{w}}^{-1}R_{\mathrm{w}} & 0 & 0 & -R_{\mathrm{w}}\Theta_{\mathrm{w}}^{-1} \\ \hline d' & \beta L' - \alpha L'R_{\mathrm{t}} & 0 & 0 & \alpha & \alpha L' & 0 \\ \hline Ld' & \begin{array}{l} -\Theta_{\mathrm{t}}^{-1}R_{\mathrm{t}} + \\ +\beta H - \alpha HR_{\mathrm{t}} \end{array} & 0 & 0 & \alpha L & \Theta_{\mathrm{t}}^{-1} + \alpha H & 0 \\ \hline 0 & 0 & 0 & -\Theta_{\mathrm{w}}^{-1}R_{\mathrm{w}} & 0 & 0 & \Theta_{\mathrm{w}}^{-1} \end{array}\right] \begin{bmatrix} A_{\mathrm{ft}} & & & & & & \\ & A_{\mathrm{t}} & & & & & \\ & & A_{\mathrm{fw}} & & & & \\ & & & A_{\mathrm{w}} & & & \\ & & & & p_{\mathrm{ft}} & & \\ & & & & & P_{\mathrm{t}} & \\ & & & & & & P_{\mathrm{w}} \end{bmatrix} \begin{bmatrix} h_{\mathrm{ft}} \\ h_{\mathrm{t}} \\ h_{\mathrm{fw}} \\ h_{\mathrm{w}} \\ \varphi_{\mathrm{ft}} \\ \varphi_{\mathrm{t}} \\ \varphi_{\mathrm{w}} \end{bmatrix}
$$

Equation (14)

transformation matrix for the two sets. This makes it possible to write (14) in terms of one set of co-ordinates to produce the required eigenvalue equations.

The absolute displacements are expressible in terms of the elastic displacements and the displacements at the reference station. Thus, as shown in Fig. 2, for the i^{th} mass in a branch of the system,

$$(h_i)=h_0-(s_i)\varphi_0+(u_i), \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$(\varphi_i)=\varphi_0+(\psi_i), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where the appropriate subscript is assigned to the terms in brackets.

The complete set of relations (15) and (16) may be expressed as:

$$\begin{bmatrix} h_0 \\ h_{ft} \\ h_t \\ h_{tw} \\ h_w \\ \varphi_{ft} \\ \varphi_t \\ \varphi_w \end{bmatrix} = \begin{bmatrix} 0 \\ u_{ft} \\ u_t \\ u_{tw} \\ u_w \\ \psi_{ft} \\ \psi_t \\ \psi_w \end{bmatrix} - \varphi_0 \begin{bmatrix} 0 & & & & & & & \\ & S_{ft} & & & & & & \\ & & S_t & & & & & \\ & & & S_{tw} & & & & \\ & & & & S_w & & & \\ & & & & & -1 & & \\ & & & & & & -I & \\ & & & & & & & -I \end{bmatrix} \begin{bmatrix} 1 \\ L \\ L \\ L \\ L \\ 1 \\ L \\ L \end{bmatrix} + h_0 \begin{bmatrix} 1 & & & & & & & \\ & I & & & & & & \\ & & I & & & & & \\ & & & I & & & & \\ & & & & I & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ L \\ L \\ L \\ L \\ 1 \\ L \\ L \end{bmatrix}$$

. . . . . (17)

In order to simplify the algebra certain matrices are now grouped together and re-defined.

Let

$$\left.\begin{aligned} &\begin{bmatrix} h_{ft} \\ h_t \\ h_{tw} \\ h_w \end{bmatrix} = h, \quad \begin{bmatrix} u_{ft} \\ u_t \\ u_{tw} \\ u_w \end{bmatrix} = u, \quad \begin{bmatrix} \varphi_{ft} \\ \varphi_t \\ \varphi_w \end{bmatrix} = \varphi, \quad \begin{bmatrix} \psi_{ft} \\ \psi_t \\ \psi_w \end{bmatrix} = \psi, \\ &\begin{bmatrix} S_{ft} & & & \\ & S_t & & \\ & & S_{tw} & \\ & & & S_w \end{bmatrix} = S, \quad \begin{bmatrix} A_{ft} & & & \\ & A_t & & \\ & & A_{tw} & \\ & & & A_w \end{bmatrix} = A, \quad \begin{bmatrix} p_{ft} & & & \\ & P_t & & \\ & & P_w & \end{bmatrix} = P, \end{aligned}\right\} \quad . \quad (18)$$

whence from (18), (17) becomes

$$\begin{bmatrix} h_0 \\ h \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ u \\ \psi \end{bmatrix} - \varphi_0 \begin{bmatrix} 0 & & \\ & S & \\ & & -I \end{bmatrix} \begin{bmatrix} 1 \\ L \\ L \end{bmatrix} + h_0 \begin{bmatrix} 1 & & \\ & I & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ L \\ L \end{bmatrix} \quad . \quad (19)$$

Section 6.1.

For equilibrium the sum of all inertia forces in the z direction is zero and the moment about any point is zero. For the moment calculated about the reference station the equations of equilibrium become respectively

$$[1\ L'\ 0]\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} h_0 \\ h \\ \varphi \end{bmatrix}=0, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$[0\ L'\ L']\begin{bmatrix} 0 & & \\ & S & \\ & & -I \end{bmatrix}\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} h_0 \\ h \\ \varphi \end{bmatrix}=0. \quad . \quad . \quad (21)$$

Substitution from (19) in (20) and (21) yields respectively

$$[1\ L'\ 0]\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} 0 \\ u \\ \psi \end{bmatrix}-\varphi_0[1\ L'\ 0]\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} 0 & & \\ & S & \\ & & -I \end{bmatrix}\begin{bmatrix} 1 \\ L \\ L \end{bmatrix}+$$

$$+h_0[1\ L'\ 0]\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} 1 & & \\ & I & \\ & & 0 \end{bmatrix}\begin{bmatrix} 1 \\ L \\ L \end{bmatrix}=0,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$[0\ L'\ L']\begin{bmatrix} 0 & & \\ & S & \\ & & -I \end{bmatrix}\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} 0 \\ u \\ \psi \end{bmatrix}-$$

$$-\varphi_0[0\ L'\ L']\begin{bmatrix} 0 & & \\ & S & \\ & & -I \end{bmatrix}\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} 0 & & \\ & S & \\ & & -I \end{bmatrix}\begin{bmatrix} 1 \\ L \\ L \end{bmatrix}+$$

$$+h_0[0\ L'\ L']\begin{bmatrix} 0 & & \\ & S & \\ & & -I \end{bmatrix}\begin{bmatrix} m_0 & & \\ & A & \\ & & P \end{bmatrix}\begin{bmatrix} 1 & & \\ & I & \\ & & 0 \end{bmatrix}\begin{bmatrix} 1 \\ L \\ L \end{bmatrix}=0. \quad . \quad (23)$$

Equations (22) and (23) are a set of two simultaneous equations in the variables h_0 and φ_0 which are soluble in terms of the elastic displacements, *i.e.* the elements of u and ψ.

It is evident that the rows and columns corresponding to m_0 in the products involving u and ψ can be omitted since they make no contribution to these terms.

The equations then respectively reduce to*

$$(L'ASL)\ddot{\varphi}_0-(m_0+L'AL)h_0=[L'\ 0]\begin{bmatrix} A & \\ & P\end{bmatrix}\begin{bmatrix} u \\ \psi\end{bmatrix} \quad . \quad . \quad (24)$$

and

$$(L'SASL+L'PL)\ddot{\varphi}_0-(L'SAL)h_0=[L'\ L']\begin{bmatrix} S & \\ & -I\end{bmatrix}\begin{bmatrix} A & \\ & P\end{bmatrix}\begin{bmatrix} u \\ \psi\end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The solution of these equations is

$$h_0=\xi[L'\ L']\begin{bmatrix} S & \\ & -I\end{bmatrix}\begin{bmatrix} A & \\ & P\end{bmatrix}\begin{bmatrix} u \\ \psi\end{bmatrix}-\sigma[L'\ 0]\begin{bmatrix} A & \\ & P\end{bmatrix}\begin{bmatrix} u \\ \psi\end{bmatrix}, \quad (26)$$

and

$$\varphi_0=\rho[L'\ L']\begin{bmatrix} S & \\ & -I\end{bmatrix}\begin{bmatrix} A & \\ & P\end{bmatrix}\begin{bmatrix} u \\ \psi\end{bmatrix}-\xi[L'\ 0]\begin{bmatrix} A & \\ & P\end{bmatrix}\begin{bmatrix} u \\ \psi\end{bmatrix}, \quad (27)$$

where
$$\left.\begin{aligned} \xi&=(L'ASL)/\gamma,\\ \sigma&=(L'SASL+L'PL)/\gamma,\\ \rho&=(m_0+L'AL)/\gamma,\\ \gamma&=(m_0+L'AL)(L'SASL+L'PL)-(L'ASL)^2,\end{aligned}\right\} \quad . \quad . \quad (28)$$

Section 6.2.

Substitution for h_0 and φ_0 from (26) and (27) in (19) then yields

$$\begin{bmatrix} h \\ \varphi\end{bmatrix}=\left[\begin{bmatrix} I & \\ & I\end{bmatrix}-\rho\begin{bmatrix} S & \\ & -I\end{bmatrix}\begin{bmatrix} L \\ L\end{bmatrix}[L'\ L']\begin{bmatrix} S & \\ & -I\end{bmatrix}\begin{bmatrix} A & \\ & P\end{bmatrix}+\right.$$
$$+\xi\begin{bmatrix} S & \\ & -I\end{bmatrix}\begin{bmatrix} L \\ L\end{bmatrix}[L'\ 0]\begin{bmatrix} A & \\ & P\end{bmatrix}+\xi\begin{bmatrix} I & \\ & 0\end{bmatrix}\begin{bmatrix} L \\ L\end{bmatrix}[L'\ L']\begin{bmatrix} S & \\ & -I\end{bmatrix}\begin{bmatrix} A & \\ & P\end{bmatrix}-$$
$$\left.-\sigma\begin{bmatrix} I & \\ & 0\end{bmatrix}\begin{bmatrix} L \\ L\end{bmatrix}[L'\ 0]\begin{bmatrix} A & \\ & P\end{bmatrix}\right]\begin{bmatrix} u \\ \psi\end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where the square matrix on the right hand side is the matrix of co-ordinate transformation.

---

*The coefficients of the variables in (24) and (25) have the following physical interpretation:

$(m_0+L'AL)\equiv$ total mass of system,

$(L'ASL)\equiv$ first moment of mass about reference station,

and $(L'SASL+L'PL)\equiv$ second moment of mass about reference station.

It is now possible to express (14) in terms of the one set of co-ordinates $\{ u, \psi \}$ * and hence to obtain the required eigenvalue equations. In (14) the absolute displacement column $\{ h, \varphi \}$ is pre-multiplied by the inertia matrix corresponding to $\{ h, \varphi \}$. On substitution of (29) this inertia matrix is modified by post-multiplication by the transformation matrix of (29), and the resulting product is the generalised inertia matrix corresponding to the co-ordinates $\{ u, \psi \}$.

If B denote this generalised inertia matrix, it is defined as

$$\left[B \right] = \left[A \quad P \right] \times \text{the transformation matrix,}$$

or

$$\begin{aligned}\left[B\right] = &\left[A \quad P\right] - \rho\left[A \quad P\right]\left[S \quad -I\right]\left[H\right]\left[S \quad -I\right]\left[A \quad P\right]' + \\ &+ \xi\left[\left[A \quad P\right]\left[S \quad -I\right]\left[H\right]\left[A \quad 0\right] + \left[A \quad 0\right]\left[H\right]\left[S \quad -I\right]\left[A \quad P\right]\right] - \\ &- \sigma\left[A \quad 0\right]\left[H\right]\left[A \quad 0\right], \qquad \ldots \qquad (30)\end{aligned}$$

where use is made of the identities

$$\begin{bmatrix} L \\ L \end{bmatrix} \begin{bmatrix} L' & L' \end{bmatrix} \equiv \left[H \right],$$

and $$\begin{bmatrix} L \\ L \end{bmatrix} \begin{bmatrix} L' & 0 \end{bmatrix} \left[A \quad P\right] \equiv \begin{bmatrix} L \\ L \end{bmatrix} \begin{bmatrix} L' & L' \end{bmatrix} \left[A \quad 0\right] \equiv \left[H\right]\left[A \quad 0\right];$$

and H is defined in Section 3.5.

The eigenvalue equations for symmetric vibration are then of the form:

$$\begin{bmatrix} u \\ \psi \end{bmatrix} = \omega^2 \left[D \right] \left[B \right] \begin{bmatrix} u \\ \psi \end{bmatrix}. \qquad \ldots \qquad (31)$$

where D is the flexibility matrix in (14) and B is the generalised inertia matrix (30).

7. Summary of Analysis

The preceding analysis leads to a statement of the eigenvalue equations for symmetric vibrations in a concise form. For numerical work on a particular aircraft it is necessary merely to insert the appropriate values of the physical properties in the

*Curly brackets thus { } denote a column matrix.

final equations and the problem is set up for solution without explicit consideration of dynamics. In this section the results of the analysis are summarised to provide all the necessary information for the calculation of the eigenvalue equations.

The equations are

$$\begin{bmatrix} u \\ \psi \end{bmatrix} = \omega^2 \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} u \\ \psi \end{bmatrix} \quad . \quad . \quad . \quad . \quad (31)$$

where $$u = \begin{bmatrix} u_{ft} \\ u_t \\ u_{tw} \\ u_w \end{bmatrix}, \qquad \psi = \begin{bmatrix} \psi_{ft} \\ \psi_t \\ \psi_w \end{bmatrix}$$

and the terms with subscripts are defined in Table I and Section 3.4; D is the flexibility matrix (first square matrix on right hand side of (14)); B is the generalised inertia matrix (30), and in respect to (30),

$$A = \begin{bmatrix} A_{ft} & & & \\ & A_t & & \\ & & A_{tw} & \\ & & & A_w \end{bmatrix}, \quad S = \begin{bmatrix} S_{ft} & & & \\ & S_t & & \\ & & S_{tw} & \\ & & & S_w \end{bmatrix}, \quad P = \begin{bmatrix} p_{ft} & & \\ & P_t & \\ & & P_w \end{bmatrix},$$

and the terms with subscripts are defined in Table I and Section 3.1. I is the unit matrix, H is a square matrix each element of which is unity, and ρ, ξ, σ are defined in (28).

Solution of (31) yields the natural frequencies and associated elastic displacement modes. The absolute displacement modes are then obtained from (17), Section 6, with the use of (26) – (28), Section 6.1.

8. Solution of Equations

Once set up in numerical form the equations may be solved by any convenient method. It is suggested that matrix iteration[5] or the escalator[7] method is suitable.

APPENDIX

Treatment of Swept and Cranked Wings

It has been assumed in the analysis that the wing and tail have straight flexural axes of the conventional type, but in modern aircraft this concept of elastic properties is frequently not even a good approximation to the true state of affairs. In particular, swept and cranked wings do not satisfy the conditions implicit in the assumption of a straight flexural axis. If it is assumed that chordwise sections of the wing do not deform (an assumption which is made for stressing wings with chordwise

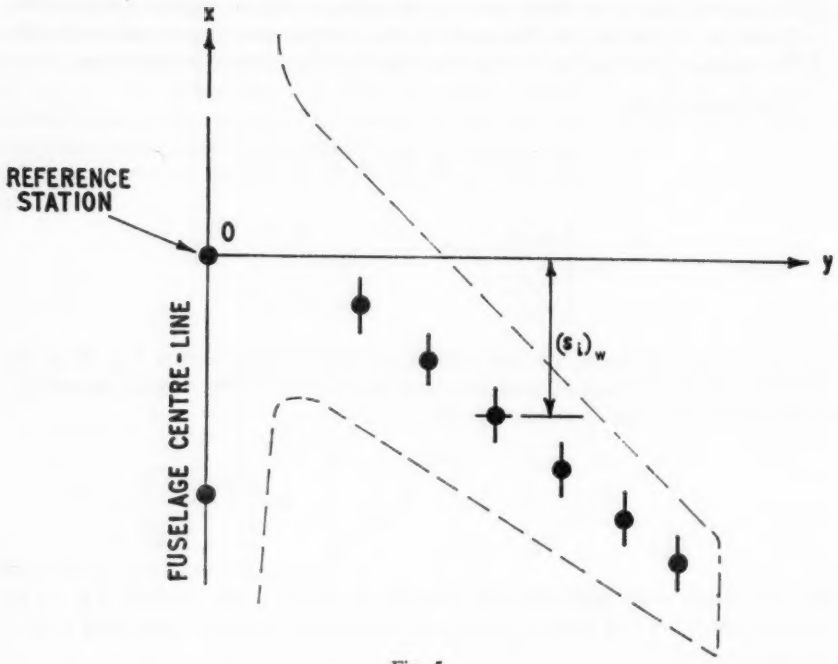

Fig. 5.
Swept wing representation.

ribs), the problems which arise are related only to the calculation of influence coefficients, while the essential form of the dynamical problem is in no way altered. In fact, if influence coefficients are measured by static test of the structure, flexural axis considerations are irrelevant.

For a wing of this type the idealised mass distribution is selected in exactly the same way as before, but no simple physical form is assumed for the elastic structure. A swept wing representation may appear as in Fig. 5. When a flexural axis exists it is assumed that the wing root forces and moments are transmitted to the fuselage at the point of intersection of the flexural axis and the fuselage centre-line. This is largely a matter of convenience since the idealised point of attachment of wing and fuselage is to some extent arbitrary; and similarly in the more general case the wing-fuselage junction point may be selected at some point within the root chord. This point then becomes the reference station on the fuselage.

It now remains to make the appropriate modifications to D of (14) to cover the more general elastic properties. To do this it is necessary merely to define certain components of sub-matrices in terms of their physical significance and to

TABLE II

Location in D		*Modification*
Row	*Column*	
4	4	$(E_w^{-1} + R_w \Theta_w^{-1} R_w)$ is replaced by matrix whose elements a_{ij} denote linear displacement of i^{th} mass of wing due to unit linear force at j^{th} mass (root encastré).
4	7	$(-R_w \Theta_w^{-1})$ is replaced by matrix whose elements a_{ij} denote linear displacement of i^{th} mass of wing due to unit torque at j^{th} spanwise station (root encastré).
7	4	$(-\Theta_w^{-1} R_w)$ is replaced by matrix whose elements a_{ij} denote angular displacement at i^{th} spanwise wing station due to unit linear force at j^{th} wing mass (root encastré) (transposed of above).
7	7	Θ_w^{-1} is replaced by matrix whose elements a_{ij} denote angular displacement at i^{th} wing station due to unit torque at j^{th} station (root encastré).

replace them by the corresponding matrices thus defined. The appropriate modifications are set out in Table II*.

Similar modifications may be necessary for matrices relating to the tail. The above table applies similarly to the tail when the subscript "t" replaces "w," "tail" replaces "wing" and the row and column numbers are ordered as follows: 2,2; 2,6; 6,2; 6,6.

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*Special attention is drawn to the sign convention for the statement of matrices in the second and third rows of Table II.

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Airscrews† at Supersonic Forward Speeds

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(*Summary*: The flow past an airscrew rotating with uniform angular velocity in a uniform supersonic stream is considered from the point of view of axes fixed in the airscrew, the motion relative to these axes being steady. A linearised equation and boundary conditions are found for the potential of the disturbance flow caused by airscrews which produce only small perturbations in the main stream. This equation is solved by expanding the potential in powers of the ratio, χ, of the tip speed of the airscrew to the speed of sound in the undisturbed stream. Equations and boundary conditions are found for the coefficients of each term of the series and it is seen that the term independent of χ satisfies the ordinary linearised potential equation for fixed axes. By subtracting suitable special integrals, the more complicated equations for the coefficients of the various powers of χ can be reduced to this ordinary potential equation. Hadamard's methods have been applied by Puckett[1], Evvard[2] and, finally, Ward[3] to the problem of finding the flow past a thin wing fixed in a supersonic stream and their methods are applied to find the terms of the series in our case. The first two terms are found explicitly for a particular type of windmill. Approximate expressions are then developed for the torque and the drag on such a windmill and the results applied in special cases. The best working conditions for windmills of this type are found by considering the effect on efficiency and power of varying the shape and the speed of rotation, within the limits imposed by the strength of the material of the airscrew.)

1. Introduction

In the supersonic motion of an aircraft or projectile through the air, it may be desirable, in some cases, to have a source of power other than the main driving force (if any). An airscrew mounted on the body and designed so that the aerodynamic reaction of the air produces a torque which tends to increase its angular velocity (*i.e.* a windmill) could be used as such a subsidiary power source. The present paper describes the flow past airscrews of this type. The theory which is developed could be modified, without any essential change, to deal with propellers; the specialisation to the case of windmills rather than to propellers is suggested

†The word airscrew is used in this paper in its general sense, referring to any type of screw designed to rotate in air. Although the theory can be modified to deal with propellers, it is applied here to windmills.—ED.

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by the greater possibility of practical application and does not relate to any limitation on the theory.

Power is obtained from the windmill at the expense of the forward momentum of the body, the reaction of the air producing a drag as well as a torque on the airscrew, and the efficiency of the windmill is measured as the ratio of the power produced to that required to overcome the drag. For high efficiency, it is desirable that the drag be as small as possible. The airscrews considered are thus taken to be thin aerofoils approximating to that aerofoil of zero thickness which, when rotating at the operational speed of the airscrew, produces no disturbance at all and therefore suffers no drag (or torque). Airscrews of this type produce only small disturbances in the fluid and it is assumed that the squares and higher powers of the perturbation velocities may be neglected. This means that linearised equations of motion may be developed and that entropy changes across any shock waves in the fluid may be neglected. It is assumed also that the effects of viscosity in the flow are confined to a negligibly thin boundary layer on the surface of the airscrew so that the flow may be considered to be isentropic. Finally, it is assumed that the air may be treated as a perfect gas, which is an excellent approximation under a wide range of conditions, including those in the atmosphere.

Airscrew theory in the subsonic case has been studied in great detail and an account of the theories which have been developed is given, for example, in Ref. 4 from which the following brief outline is, in the main, derived. References to the authors mentioned here, and many others, are given by Durand. The earliest theory, the "momentum theory" of Rankine and Froude, ignored the geometrical shape of the airscrew, replacing it by an "actuator disc" and deriving the resultant forces on the airscrew by consideration of the overall changes in the momentum of the fluid. The "blade element" theory, due initially to Froude and mainly to Drzewiecki and Lanchester did treat in detail the flow past the airscrew by dividing the blade into elements along its radius and treating each of these as an aerofoil section moving in a main stream which is modified to allow for the transverse component of velocity caused by the rotation. It was found that this theory could explain the behaviour of an airscrew in general terms only and that it failed to give accurate numerical results. Moreover, the theory suggested that the efficiency of an airscrew tends to unity as the drag tends to zero, a serious divergence from the results of the momentum theory which set an upper limit to the efficiency.

Side by side with the blade element theory for airscrews, there has developed the "quasi-static" theory for treating the flow past wings in roll. In this theory, allowance is made for the (small) rotation of the wing, not by modifying the main stream as in the blade element theory but by assuming a suitable change in the shape of the wing section (more accurately, by making an appropriate change to the boundary condition at the surface of the aerofoil). It is apparent that these two theories achieve the same results for the wing in roll. Although the quasi-static theory has the advantage in this case that it enables the problem to be solved by using the ordinary methods for finding the flow past an aerofoil at rest in a uniform stream,

the blade element theory is more general, being applicable without the restriction to slow rates of rotation.

Eventually, the key to the completion (and reconciliation) of the theories was found by Lanchester to lie in the vortices shed by the airscrew and composing its wake. These produce an interference velocity at the blades which has to be taken into consideration when modifying the incident velocity at the surface of the aerofoil. The difficulty of the "vortex theory" is in the analysis of the vortex distribution in the wake and the derivation of the correct contribution to the incident flow. Betz, Prandtl and Goldstein attacked the problem by considering the actual helical vortex sheets in the wake but a common procedure has been to simplify the problem by assuming that there is a large number of blades and postulating a uniform distribution of vorticity in the wake, thus neglecting the periodicity associated with a small number of blades.

In supersonic flow, the problem is much easier since the effects of disturbances in the fluid are confined to a region downstream from the points where they occur, so that the trailing vortices have no influence whatever on conditions on the surface of the blades of the airscrew. For this reason, too, there is no interference between the blades of the airscrew, provided that each blade lies outside the region of influence of the others. For high pitch, which would be necessary at high speeds to keep the angular velocity down to a level which the materials can stand, this condition is clearly satisfied. In the supersonic case then, the method of the blade element theory, or (as is preferable) of the quasi-static theory, may be used. However, it is now possible to go further and demonstrate that the solution obtained from this theory is the first approximation for a slowly rotating airscrew and to find the second and higher approximations.

The problem is considered from the point of view of axes fixed in the airscrew which is taken, for convenience, to be rotating with uniform angular velocity Ω about a fixed point in a uniform supersonic stream of velocity U parallel to the axis of rotation. The linearised equation for the disturbance potential is found and solved approximately for airscrews for which the ratio† χ of the tip speed Ωb to the speed of sound in the main stream is small. The potential is expanded as a power series in χ and a method given for finding the terms of the series in succession. In each case, the problem is reduced to that of finding the flow past an aerofoil fixed in a uniform supersonic stream using the methods developed by Puckett[1], Evvard[2] and Ward[3]. The first term of the series is exactly the solution derived from the quasi-static theory and the later terms give the higher approximations to the flow.

The theory is applied to a particular type of airscrew for which the first two terms of the series for the potential are worked out. Approximate expressions are then found for the drag and the torque on an airscrew of this type and the results

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† $\chi = \Omega b/a_0 = MX = \pi M/J$ where $M = U/a_0$. $X = \Omega b/U$ is the parameter defined by Durand (Ref. 4, p. 328) for windmills and J is the advance/diameter ratio defined for propellers by Durand (Ref. 4, p. 173).

applied in special cases. The power which can be derived from such airscrews and the efficiency at which they work are found for a range of values of a parameter which determines the shape of an airscrew of given plan form and thickness, rotating at given angular velocity. It is found that the efficiency reaches a maximum for a certain value of this parameter which therefore gives the best working conditions. The variation in this maximum efficiency and the corresponding power with the angular velocity is then considered. Best results are obtained for high values of the angular velocity, provided that the tip speed does not exceed a limit imposed by several restrictions, some of a physical nature and others concerned with the limitations of the linearised theory. Finally, an estimate is made of the way in which the strength of the material of the airscrew restricts the shape.

Notation

$\alpha, \beta = x \pm By$ $\alpha', \beta' = x' \pm By'$	characteristic co-ordinates for the equations for potential flow relative to fixed axes
ε	parameter determining the thickness of the airscrew
$\zeta = P/(\pi c^2 \rho_0 U^3)$	power coefficient
ζ_m	value of ζ when airscrew is working at maximum efficiency
κ, ν	coefficients depending on μ and χ and defined by $\eta = (\lambda + \kappa\varepsilon)/(\lambda^2 + \lambda + \nu\varepsilon^2)$
λ	ω/Ω
λ_m	value of λ which gives maximum efficiency
μ	b/c
ν	*see* κ, ν
ρ	density of air
ρ_0	density of air in undisturbed main stream
ϕ	perturbation potential
$\phi_{(1)}, \phi_{(2)}$	two parts of ϕ as defined in Section 3
$\phi_0, \phi_1, \phi_2, \ldots$ $\phi_{0(i)}, \phi_{1(i)}, \phi_{2(i)}, \ldots$	coefficients of powers of χ in series expansions of ϕ, $\phi_{(i)}$
$\bar{\phi}_1, \bar{\phi}_2, \ldots$	solutions of the equations for potential flow relative to fixed axes associated with $\phi_1, \phi_2, \ldots$
$\chi = \Omega b/a_0$	ratio of tip speed of airscrew to speed of sound in main stream
Ω	angular velocity of airscrew
ω	defined by the relation $\Omega + \omega =$ angular velocity at which mean surface of airscrew causes zero disturbance

a	local speed of sound
a_0	speed of sound in undisturbed main stream
b	radius of airscrew
B	$(M^2-1)^{\frac{1}{2}}$
c	chord of airscrew
D	drag
E	Young's modulus
$\mathbf{F}$	aerodynamic force on airscrew
I	quadratic moment of root section of airscrew blade
$I(\alpha, \beta)$	integral in Section 3.11
J	advance/diameter ratio for propeller—*see* footnote, p. 25
M	Mach number of undisturbed main stream $(=U/a_0)$
$\mathbf{n}$	normal to surface $z=y\tan(\Omega x/U)$
$\mathbf{n}^*$	normal to actual airscrew surface
P	power derived from airscrew $(=\Omega Q)$
Q	airscrew torque
p	pressure
Δp	perturbation pressure
p_0	pressure in undisturbed main stream
$\mathbf{r}, \mathbf{r}'$	position vectors
S	surface of integration in Section 4
S'	maximum stress in airscrew blade in Section 4.1
$\mathbf{U}$	velocity of main stream relative to fixed axes
$\mathbf{U}'$	velocity of main stream relative to moving axes
$\mathbf{u}$	perturbation velocity
$\mathbf{v}$	stream velocity relative to fixed axes
$\mathbf{v}'$	stream velocity relative to moving axes
$\mathbf{w}$	perturbation mass flux vector
X	ratio of tip speed of airscrew to main stream velocity $(=\Omega b/U)$, *see* footnote, p. 25
x, y, z x', y', z'	components of position vectors

2. Mathematical Formulation

Let x, y, z be a right handed system of Cartesian co-ordinates such that x is measured in the direction of the main stream, which has velocity U relative to the axes. Let the axes be fixed in the airscrew so that they are rotating with constant angular velocity $\mathbf{\Omega}=(-\Omega, 0, 0)$, the angular velocity being positive in the direction of forward motion of the airscrew. Since the airscrew is constantly at rest relative to these rotating axes, the motion referred to them is steady.

Let $\mathbf{v}$ be the fluid velocity relative to fixed axes with which the rotating axes instantaneously coincide and let $\mathbf{v}'$ be the velocity relative to the moving axes; then

$$\mathbf{v}=\mathbf{v}'+\mathbf{\Omega}\wedge\mathbf{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the motion is assumed to be irrotational relative to the fixed axes, then

$$\text{curl } \mathbf{v}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Bernoulli's equation for the pressure in steady motion is (Ref. 5, p. 20)

$$\frac{1}{2}\mathbf{v}^2-\mathbf{v}.\mathbf{\Omega}\wedge\mathbf{r}+\int dp/\rho=\text{constant}. \quad . \quad . \quad . \quad (3)$$

The equation of continuity in steady motion is (Ref. 5, p. 12)

$$\text{div } (\rho\mathbf{v}')=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equations (1) to (4) are the equations of motion for the steady irrotational flow past the airscrew.

Let $\mathbf{v}=\mathbf{U}+\mathbf{u}$ . . . . . (5)

where, by (2), $\mathbf{u}=\text{grad}\,\phi$; . . . . . (6)

ϕ is the perturbation velocity potential. It is assumed that the disturbances caused by the airscrew are sufficiently small for terms quadratic in ϕ to be neglected.

Let p_0, ρ_0 and a_0 be the pressure, density and speed of sound in the undisturbed stream (where $\mathbf{v}=\mathbf{U}$). Then, from Bernoulli's equation (3),

$$\int_{p_0}^{p}\frac{dp}{\rho}=-\mathbf{u}.\mathbf{U}'-\frac{1}{2}\mathbf{u}^2, \quad . \quad . \quad . \quad . \quad (7)$$

where $\mathbf{U}'=\mathbf{U}-\mathbf{\Omega}\wedge\mathbf{r}$ . . . . . (8)

is the main stream velocity relative to the rotating axes. Equations (1), (5) and (8) show that

$$\mathbf{v}' = \mathbf{U}' + \mathbf{u}. \qquad (9)$$

By expanding the integral on the left hand side of (7), the pressure and density are obtained in terms of the velocity in the forms

$$(p - p_0)/p_0 = -\mathbf{U}' \cdot \mathbf{u} - \tfrac{1}{2}\mathbf{u} \cdot \mathbf{w} + O|\mathbf{u}|^3, \qquad (10)$$

where
$$\mathbf{w} = \mathbf{u} - (\mathbf{U}' \cdot \mathbf{u})\mathbf{U}'/a_0^2, \qquad (11)$$

and
$$\rho = \rho_0 (1 - \mathbf{U}' \cdot \mathbf{u}/a_0^2) + O|\mathbf{u}|^2. \qquad (12)$$

The vector **w** can be interpreted as a perturbation mass flux vector in the sense that

$$\rho \mathbf{v}' = \rho_0 (\mathbf{U}' + \mathbf{w}) + O|\mathbf{u}|^2. \qquad (13)$$

Finally, substituting for $\rho \mathbf{v}'$ from equation (13) in the continuity equation (4) gives, correct to $O(|\mathbf{u}|)$,

$$\operatorname{div} \mathbf{w} = 0. \qquad (14)$$

By use of equations (6), (8) and (11), equation (14) becomes the equation for the disturbance potential ϕ, namely

$$(1 - M^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\Omega^2}{a_0^2}\left(z^2 \frac{\partial^2 \phi}{\partial y^2} - 2yz \frac{\partial^2 \phi}{\partial y \partial z} + y^2 \frac{\partial^2 \phi}{\partial z^2} - y \frac{\partial \phi}{\partial y} - z \frac{\partial \phi}{\partial z}\right) + \\ + \frac{2\Omega U}{a_0^2}\left(y \frac{\partial^2 \phi}{\partial z \partial x} - z \frac{\partial^2 \phi}{\partial x \partial y}\right), \qquad (15)$$

where $M = U/a_0$.

In cylindrical polar co-ordinates x, r, θ for which

$$y = r\cos\theta, \quad z = r\sin\theta,$$

the equation is

$$(1 - M^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{\Omega^2}{a_0^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{2\Omega U}{a_0^2}\frac{\partial^2 \phi}{\partial x \partial \theta} \qquad (16)$$

Equations (15) and (16) will be satisfied by the disturbance potential of the flow past an airscrew rotating with angular velocity $(-\Omega, 0, 0)$ in a uniform main stream with velocity (U, 0, 0), provided the perturbations are small. It has already been pointed out that for this to be so, the airscrew must approximate to the surface (of zero

thickness) which would produce no disturbance at all when rotating in the same way. The normal $\mathbf{n}$ to such a surface must satisfy the condition

$$\mathbf{U}'.\mathbf{n}=0, \qquad . \quad . \quad . \quad . \quad . \quad (17)$$

where $\mathbf{U}'$ is given by (8) as the velocity of the undisturbed main stream relative to the axes fixed in the rotating surface. If the surface has equation $f(x, y, z)=0$ relative to these axes, condition (17) gives as the equation for $f(x, y, z)$:

$$U\frac{\partial f}{\partial x}-\Omega z\frac{\partial f}{\partial y}+\Omega y\frac{\partial f}{\partial z}=0. \qquad . \quad . \quad . \quad . \quad (18)$$

The surface which satisfies this equation and passes through the line $x=z=0$ has equation

$$z=y\tan(\Omega x/U) \qquad . \quad . \quad . \quad . \quad . \quad (19)$$

and the equation of the airscrew surface is assumed to approximate to (19).

The boundary condition on the surface of the airscrew is, exactly as in (17), that the normal component of the fluid velocity relative to the airscrew be zero,

i.e. $$\mathbf{v}'.\mathbf{n}^*=0, \qquad . \quad . \quad . \quad . \quad . \quad (20)$$

where $\mathbf{n}^*$ is the normal to the surface and, since the surface differs only slightly from (19), can be taken as differing from $\mathbf{n}$ by a vector whose magnitude is $O|\mathbf{u}|$. Substituting for $\mathbf{v}'$ from equation (9) gives finally as the fully linearised boundary condition that

$$\partial\phi/\partial n=-\mathbf{U}'.\mathbf{n}^*, \qquad . \quad . \quad . \quad . \quad (21)$$

which, to a first order approximation, may be applied on the surface $z=y\tan(\Omega x/U)$ over the plan form of the airscrew.

The complete mathematical problem to be solved is given by equations (15) or (16) under boundary condition (21), subject to the vanishing of the disturbance on any plane, normal to the flow, which lies ahead of the airscrew.

3. Solution of the Potential Equation in Series

The equations and boundary conditions formulated in Section 2 apply to all airscrews, whether propellers or windmills, but in developing the method of solution, it is convenient to consider a particular type of airscrew and, for the reasons indicated in the introduction, a windmill is chosen.

If an airscrew of zero thickness whose surface is given by equation (19) rotates at angular velocity Ω, it causes no disturbance in the air and there are no forces

acting on it. If now the speed of rotation is reduced slightly, there will be small disturbances in the air and the reaction of the air will produce a torque which will tend to restore the angular velocity to the value Ω. If this torque is utilised in doing work and the velocity maintained at a value less than Ω, the airscrew is being used as a windmill to provide power, the amount of which depends on the extent to which the angular velocity is reduced.

Accordingly, an airscrew is considered which has its mean surface given by

$$z = y \tan [(\Omega + \omega) x/U] \quad . \quad . \quad . \quad . \quad (22)$$

(where ω is small), but which rotates at speed Ω, so that there will be a torque acting on it and tending to increase the angular velocity. It is necessary to give the airscrew a finite thickness and the upper and lower surfaces of the airscrew are taken to be given by the equations,

$$z = y \tan [(\Omega + \omega) x/U] \pm \varepsilon f(x, y) \quad . \quad . \quad . \quad (23)$$

(where ε is small). For convenience, the symmetrical case only is considered but it may be seen that no difficulty of principle arises in the unsymmetrical case. If propellers rather than windmills were being discussed, the equations of the surfaces of the propeller would be the same as (23), except that $(\Omega + \omega)$ would be replaced by $(\Omega - \omega)$.

The potential equation (15) is now solved, with boundary condition (21), for the airscrew given by equation (23). Consider first the upper surface (23) and suppose that squares and higher powers of ε and ω may be neglected. Applying boundary condition (21) to this surface will lead to a condition of the form

$$(\partial\phi/\partial n)_{z = y \tan (\Omega x/U) + 0} = \omega\, g_1(x, y) + \varepsilon\, g_2(x, y) . \quad . \quad . \quad (24)$$

Similarly, applying (21) to the lower surface (23) leads to the condition

$$(\partial\phi/\partial n)_{z = y \tan (\Omega x/U) - 0} = \omega g_1(x, y) - \varepsilon\, g_2(x, y) \quad . \quad . \quad (25)$$

ϕ can thus be split into two parts $\phi_{(1)}$ and $\phi_{(2)}$ which satisfy boundary conditions:

$$(\partial\phi_{(1)}/\partial n)_{z = y \tan (\Omega x/U) + 0} = (\partial\phi_{(1)}/\partial n)_{z = y \tan (\Omega x/U) - 0} = \omega\, g_1(x, y) \quad . \quad (26)$$

and

$$(\partial\phi_{(2)}/\partial n)_{z = y \tan (\Omega x/U) + 0} = -(\partial\phi_{(2)}/\partial n)_{z = y \tan (\Omega x/U) - 0} = \varepsilon\, g_2(x, y) . . \quad (27)$$

It is easily seen that $\phi_{(1)}$ is the potential for the flow past the aerofoil whose equation is obtained by putting ε zero in equation (23) and is thus given by (22). Similarly, $\phi_{(2)}$ is the potential for the flow past the aerofoil whose equation is obtained by putting ω zero in (23) and so is

$$z = y \tan (\Omega x/U) \pm \varepsilon f(x, y) . \quad . \quad . \quad . \quad . \quad (28)$$

Methods for finding $\phi_{(1)}$ and $\phi_{(2)}$ are given in Sections 3.1 and 3.2.

3.1. *Airscrew of zero thickness designed to rotate at angular velocity* $(\Omega+\omega)$ *without disturbing the flow*

In finding the first part $\phi_{(1)}$ of the potential for the flow past the airscrew with surfaces given by equations (23), the flow must be found past the aerofoil of zero thickness which rotates at angular velocity Ω but is designed to rotate without disturbing the flow at angular velocity $(\Omega+\omega)$ and so is given by equation (22).

In theory, this problem can be solved directly by means of Hadamard's general theory for boundary value problems for hyperbolic equations, using methods analogous to those for the ordinary problem of an aerofoil fixed in a uniform stream. Attempts to do this were unsuccessful because of the complexity of the integrals involved. However if the potential ϕ is expanded in powers of the ratio of the tip speed, Ωb, to the speed of sound in the main stream, a_0, as

$$\phi=\phi_0+(\Omega b/a_0)\phi_1+(\Omega b/a_0)^2\phi_2+\ldots., \qquad (29)$$

Hadamard's method can be applied to find each of the terms ϕ_0, ϕ_1, ... in turn. In fact, the problem to be solved in each case reduces to solving the equation

$$(1-M^2)\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2=0 \qquad (30)$$

for the flow past an aerofoil fixed in a uniform main stream. Hadamard's method has been used to solve this problem in detail and a complete account of this is given by Ward[3].

The procedure is to substitute for ϕ from (29) in the equation (15) and boundary condition (21) and, by equating coefficients of powers of $\Omega b/a_0$, to obtain the equations and boundary conditions satisfied by ϕ_0, ϕ_1, . . . This leads to the following equations for the terms of the series

$$(1-M^2)\frac{\partial^2\phi_0}{\partial x^2}+\frac{\partial^2\phi_0}{\partial y^2}+\frac{\partial^2\phi_0}{\partial z^2}=0; \qquad (31)$$

$$(1-M^2)\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}+\frac{\partial^2\phi_1}{\partial z^2}=\frac{2M}{b}\left(y\frac{\partial^2\phi_0}{\partial z\,\partial x}-z\frac{\partial^2\phi_0}{\partial x\,\partial y}\right); \qquad (32)$$

$$(1-M^2)\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial y^2}+\frac{\partial^2\phi_2}{\partial z^2}=\frac{2M}{b}\left(y\frac{\partial^2\phi_1}{\partial z\,\partial x}-z\frac{\partial^2\phi_1}{\partial x\,\partial y}\right)+\frac{1}{b^2}\left(z^2\frac{\partial^2\phi_0}{\partial y^2}-\right.$$
$$\left.-2yz\frac{\partial^2\phi_0}{\partial y\,\partial z}+y^2\frac{\partial^2\phi_0}{\partial z^2}-y\frac{\partial\phi_0}{\partial y}-z\frac{\partial\phi_0}{\partial z}\right). \qquad (33)$$

Only ϕ_0 and ϕ_1 are considered in what follows, and, for determining the boundary conditions on these, Ω and ω may be assumed sufficiently small for ω^2

and Ω^3 (and hence, also, for example, $\Omega^2\omega$ and higher powers) to be neglected. In this case, the direction cosines of the normal $\mathbf{n}^*$ to the surface (22) are

$$-(\Omega+\omega)y/U\,,\quad -(\Omega+\omega)x/U\,,\quad 1-(\Omega^2+2\omega\Omega)(x^2+y^2)/(2U^2)\,. \qquad (34)$$

It follows that the boundary condition (21) becomes

$$\partial\phi/\partial n=\omega y \text{ on } z=y\tan(\Omega x/U) \qquad (35)$$

to the order of the approximation and, as (26) indicates, this condition holds on both sides of the surface. An expansion of $(\partial\phi/\partial n)_{z=y\tan(\Omega x/U)}$ by means of Taylor's theorem gives

$$\left(\frac{\partial\phi}{\partial n}\right)_{z=y\tan(\Omega x/U)}=\left(\frac{\partial\phi}{\partial n}\right)_{z=0}+\frac{\Omega xy}{U}\left(\frac{\partial^2\phi}{\partial z\,\partial n}\right)_{z=0} \qquad (36)$$

to the order of the approximation previously defined; and since $\partial\phi/\partial n$ is the component of grad ϕ along the normal to the surface $z=y\tan(\Omega x/U)$,

$$\frac{\partial\phi}{\partial n}=-\frac{\Omega y}{U}\frac{\partial\phi}{\partial x}-\frac{\Omega x}{U}\frac{\partial\phi}{\partial y}+\left[1-\frac{\Omega^2}{2U^2}(x^2+y^2)\right]\frac{\partial\phi}{\partial z}. \qquad (37)$$

Substituting for ϕ from (29) and combining the results of (36) and (37), from the boundary condition (35) are obtained the following boundary conditions on ϕ_0 and ϕ_1, over the plan form:

$$(\partial\phi_0/\partial z)_{z=\pm0}=\omega y\,, \qquad (38)$$

$$\left(\frac{\partial\phi_1}{\partial z}\right)_{z=\pm0}=\frac{1}{Mb}\left(x\frac{\partial\phi_0}{\partial y}+y\frac{\partial\phi_0}{\partial x}-xy\frac{\partial^2\phi_0}{\partial z^2}\right)_{z=\pm0}. \qquad (39)$$

To obtain the conditions on ϕ_2, ϕ_3, . . . it would be necessary to retain terms in the cube and higher powers of Ω in forming condition (35). The condition on ϕ_2, for example, is

$$\left(\frac{\partial\phi_2}{\partial z}\right)_{z=\pm0}=\frac{\omega y}{2M^2b^2}(3x^2-y^2)+\frac{1}{Mb}\left(x\frac{\partial\phi_1}{\partial y}+y\frac{\partial\phi_1}{\partial x}-xy\frac{\partial^2\phi_1}{\partial z^2}\right)_{z=\pm0}+$$
$$+\frac{xy}{M^2b^2}\left(\frac{x^2+y^2}{2xy}\frac{\partial\phi_0}{\partial z}+y\frac{\partial^2\phi_0}{\partial z\,\partial x}+x\frac{\partial^2\phi_0}{\partial y\,\partial z}-\frac{xy}{2}\frac{\partial^3\phi_0}{\partial z^3}\right)_{z=\pm0} \qquad (40)$$

The equation (31) and boundary condition (38) satisfied by ϕ_0 are identical with those given for the potential by the quasi-static theory. On this theory the airscrew is treated as an aerofoil fixed in a steady stream and the (small) effect of

the rotation is allowed for by amending the boundary condition at the surface of the airscrew. This means that the potential is given by equation (30), the ordinary linearised equation in fixed axes, but the boundary condition becomes

$$(\partial\phi/\partial z)_{z=0}=(-\mathbf{U}'\,.\,\mathbf{n}^*)_{z=0}; \qquad . \quad . \quad . \quad . \quad (41)$$

in the case of a fixed aerofoil in a uniform stream, the right hand side of (41) would be $(-\mathbf{U}\,.\,\mathbf{n}^*)_{z=0}$. For the particular airscrew under consideration, (41) reduces to

$$(\partial\phi/\partial z)_{z=0}=\omega y,$$

which is the same as condition (38) for ϕ_0. Thus the quasi-static theory gives the first approximation to the flow past the airscrew for small angular velocities. The method has been used, for example by Robinson[6], in the case of a delta wing.

The form of the boundary condition (38) on $\partial\phi_0/\partial z$ shows that ϕ_0 is an odd function of z and, since equation (31) is the usual linearised potential equation, ϕ_0 can be found by the method given by Ward[3] for the anti-symmetrical part of the potential for the flow past a thin aerofoil fixed in a uniform supersonic stream.

The first step in solving equation (32) for ϕ_1 with boundary condition (39) is to find a particular integral of the equation. Using this integral it is then possible to reduce the problem again to the solution of the ordinary equation for a fixed aerofoil, this time in the symmetrical case treated by Puckett[1] (and also given by Ward[3]). In cylindrical polar co-ordinates, equation (32) becomes

$$(1-M^2)\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2}=\frac{2M}{b}\frac{\partial^2\phi_0}{\partial x\,\partial\theta}, \qquad . \quad . \quad (42)$$

and the expression

$$\phi_1=\frac{Mx}{b(1-M^2)}\frac{\partial\phi_0}{\partial\theta}=\frac{Mx}{b(1-M^2)}\left(y\frac{\partial\phi_0}{\partial z}-z\frac{\partial\phi_0}{\partial y}\right) \qquad . \quad . \quad (43)$$

can be shown, by substituting in the equation, to be a particular integral. Thus

$$\phi_1=\bar{\phi}_1+\frac{Mx}{b(1-M^2)}\left(y\frac{\partial\phi_0}{\partial z}-z\frac{\partial\phi_0}{\partial y}\right) \qquad . \quad . \quad . \quad (44)$$

provided that $\bar{\phi}_1$ satisfies the equation

$$(1-M^2)\frac{\partial^2\bar{\phi}_1}{\partial x^2}+\frac{\partial^2\bar{\phi}_1}{\partial y^2}+\frac{\partial^2\bar{\phi}_1}{\partial z^2}=0. \qquad . \quad . \quad . \quad (45)$$

Substituting for ϕ_1 from (44) in boundary condition (39) gives as the condition on $\bar{\phi}_1$ over the plan form:

$$\left(\frac{\partial\bar{\phi}_1}{\partial z}\right)_{z=\pm0}=\frac{1}{Mb}\left[y\frac{\partial\phi_0}{\partial x}+\frac{1}{M^2-1}\left(xy\frac{\partial^2\phi_0}{\partial z^2}-x\frac{\partial\phi_0}{\partial y}\right)\right]_{z=\pm0}. \qquad . \quad (46)$$

Since ϕ_0 is an odd function of z, this condition is such that

$$(\partial\bar{\phi}_1/\partial z)_{z=+0}=(\partial\bar{\phi}_1/\partial z)_{z=-0} \quad . \quad . \quad . \quad . \quad (47)$$

so that $\bar{\phi}_1$ is an even function of z and can be found by Puckett's method.

In the same way, the problem of finding $\phi_2, \phi_3, \ldots$ can be reduced, by finding particular integrals of the equations, to that of finding, alternately, the symmetrical or the anti-symmetrical part of the potential past a fixed aerofoil in a uniform supersonic stream. For example,

$$\phi_2=\bar{\phi}_2-\frac{Mx}{bB^2}\frac{\partial\bar{\phi}_1}{\partial\theta}+\tfrac{1}{2}\left(\frac{Mx}{bB^2}\right)^2\frac{\partial^2\phi_0}{\partial x^2}+\frac{x}{2B^4b^2}\int_0^x\frac{\partial^2\phi_0}{\partial\theta^2}dx', \quad . \quad . \quad (48)$$

where $B^2=M^2-1$

and $$(1-M^2)\frac{\partial^2\bar{\phi}_2}{\partial x^2}+\frac{\partial^2\bar{\phi}_2}{\partial y^2}+\frac{\partial^2\bar{\phi}_2}{\partial z^2}=0;$$

and $\partial\bar{\phi}_2/\partial z$ is given by substituting for ϕ_2 from (48) in condition (40) for $(\partial\phi_2/\partial z)_{z=\pm0}$.

3.1.1. *Solution for ϕ_0 for an aerofoil of rectangular plan form and independent wing tips*

ϕ_0 satisfies equation (31):

$$B^2\frac{\partial^2\phi_0}{\partial x^2}-\frac{\partial^2\phi_0}{\partial y^2}-\frac{\partial^2\phi_0}{\partial z^2}=0,$$

and boundary condition (38) on the plan form:

$$(\partial\phi_0/\partial z)_{z=\pm0}=\omega y.$$

Solving for ϕ_0 in the particular case of rectangular plan form, let $ABCD$ (Fig. 1) be the plan form with span $AD=2b$ and chord $AB=c$. The origin of axes is the mid-point of the leading edge with the x-axis in the direction of the stream and the y-axis along the leading edge. Let AF and DE be the characteristic lines for equation (31) through A and D respectively and lying inside the plan form.

Then, from Ward's[3] equation (13), at a point $P(x, y, 0)$ inside $AFED$,

$$\phi_0(x,y,0)=-\frac{\omega}{\pi}\int\int\frac{y'\,dx'\,dy'}{\{(x-x')^2-B^2(y-y')^2\}^{\frac{1}{2}}}, \quad . \quad . \quad (49)$$

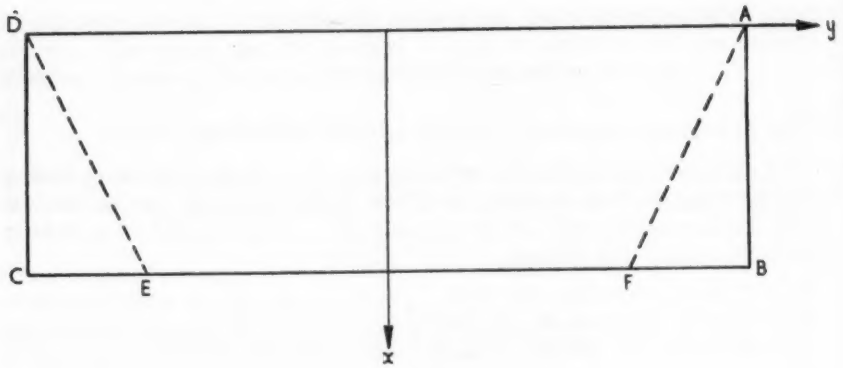

Fig. 1.
Plan form of airscrew.

where the integration is taken over the area of the plane $z=0$ for which $x' \leqslant x \pm B(y-y')$. Following Ward, the integral in (49) is transformed by introducing characteristic co-ordinates (a method due to Evvard):

$$\alpha = x - By, \quad \beta = x + By, \qquad . \quad . \quad . \quad . \quad (50)$$

and the equation

$$\phi_0 = \frac{\omega}{4B^2\pi} \int\int \frac{(\alpha'-\beta')\,d\alpha'\,d\beta'}{\{(\alpha-\alpha')(\beta-\beta')\}^{\frac{1}{2}}} \qquad . \quad . \quad . \quad (51)$$

is obtained, the integration being taken over the area $\alpha' \leqslant \alpha$, $\beta' \leqslant \beta$. The integral in (51) is split into two parts so that

$$\phi_0 = \frac{\omega}{4B^2\pi}\Big[I(\alpha,\beta) - I(\beta,\alpha)\Big] \qquad . \quad . \quad . \quad . \quad (52)$$

where

$$I(\alpha,\beta) = \int_a^{-\beta} \frac{\alpha'\,d\alpha'}{(\alpha-\alpha')^{\frac{1}{2}}} \int_\beta^{-\alpha'} \frac{d\beta'}{(\beta-\beta')^{\frac{1}{2}}}$$

$$= \frac{\pi}{4}(\alpha+\beta)(3\alpha-\beta). \qquad . \quad . \quad . \quad . \quad (53)$$

Then, from (52) and (53),

$$\phi_0 = -\frac{\omega}{4B^2}(\beta^2-\alpha^2) = -\frac{\omega xy}{B}. \qquad . \quad . \quad . \quad . \quad (54)$$

Again, for a point $P(x, y, 0)$ on the wing tip ABF, from Ward's[3] equation (30), ϕ_0 is given by an integral of the same form as (51) but over the area $KLPM$ in Fig. 2. The integral can be split into three parts and eventually gives

$$\phi_0(x, y, 0) = -\frac{2\omega}{B^2\pi}\{ B\,xy \sin^{-1}[B(b-y)/x]^{\frac{1}{2}} - \tfrac{1}{3}[2x - B(y+2b)] \times \times [B(b-y)(x+B(y-b))]^{\frac{1}{2}} \}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

A similar result can be obtained for the wing tip CDE.

ϕ_0 has thus been found at all points on the surface of the aerofoil. It may be noted that ϕ_0 and $\partial\phi_0/\partial x$ are both continuous across the line AF and across the wing tip AB ($\phi_0 \equiv 0$ for $|y| > b$).

3.1.2. *Solution for ϕ_1 for an aerofoil with rectangular plan form and independent wing tips*

As has already been shown, the problem of finding ϕ_1 reduces to that of finding $\bar{\phi}_1$ which is given by equation (45):

$$B^2\frac{\partial^2\bar{\phi}_1}{\partial x^2} - \frac{\partial^2\bar{\phi}_1}{\partial y^2} - \frac{\partial^2\bar{\phi}_1}{\partial z^2} = 0,$$

with boundary condition (46) on the plan form:

$$\left(\frac{\partial\bar{\phi}_1}{\partial z}\right)_{z=0} = \frac{1}{Mb}\left[y\frac{\partial\phi_0}{\partial x} + \frac{1}{B^2}\left(xy\frac{\partial^2\phi_0}{\partial z^2} - x\frac{\partial\phi_0}{\partial y}\right)\right]_{z=0}.$$

Since $\bar{\phi}_1$ is an even function of z,

$$(\partial\bar{\phi}_1/\partial z)_{z=0} = 0$$

at points of the plane $z=0$ outside the plan form, and $\bar{\phi}_1(x, y, 0)$ is given, at all points of the plane, by

$$\bar{\phi}_1(x, y, 0) = -\frac{1}{\pi}\int\int\left[\frac{\partial\bar{\phi}_1(x', y', z')}{\partial z'}\right]_{z'=0}\frac{dx'\,dy'}{\{(x-x')^2 - B^2(y-y')^2\}^{\frac{1}{2}}} \quad . \quad (56)$$

the integration being over the area for which

$$x' \leqslant x \pm B(y-y').$$

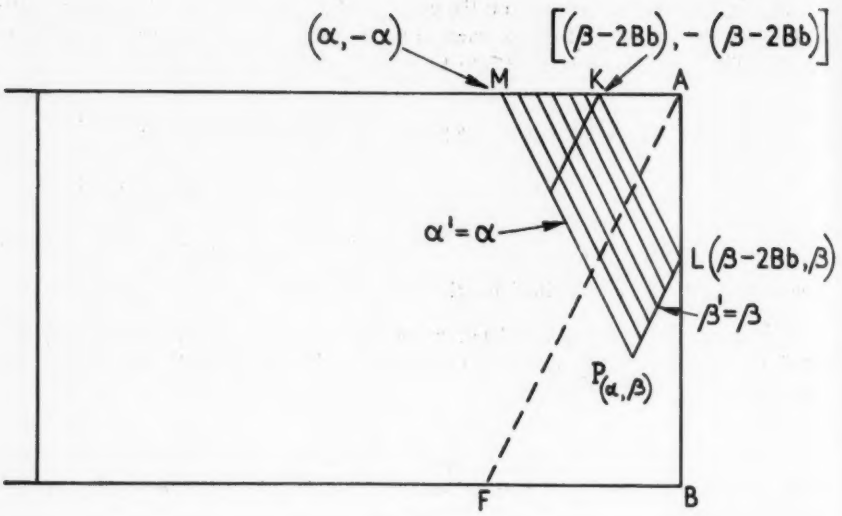

Fig. 2.

Now, for points in *AFED*,

$$(\partial\bar{\phi}_1/\partial z)_{z=0} = -\omega\,(x^2 - B^2\,y^2)/(Mb\,B^3), \quad . \quad . \quad . \quad (57)$$

so that for points $P(x, y, 0)$ in *AFED*,

$$\bar{\phi}_1(x, y, 0) = \frac{\omega}{Mb\,B^3\,\pi}\int\int \frac{(x'^2 - B^2\,y'^2)\,dx'\,dy'}{\{(x-x')^2 - B^2\,(y-y')^2\}^{\frac{1}{2}}}\,dx'\,dy',$$

or, in characteristic co-ordinates,

$$\bar{\phi}_1 = \frac{-\omega}{2\,Mb\,B^4\,\pi}\int_{\alpha}^{-\beta}\frac{\alpha'\,d\alpha'}{(\alpha-\alpha')^{\frac{1}{2}}}\int_{\beta}^{-\alpha'}\frac{\beta'\,d\beta'}{(\beta-\beta')^{\frac{1}{2}}}$$

$$= \frac{\omega\,(\alpha+\beta)}{48\,Mb\,B^4}(5\,\alpha^2 - 14\,\alpha\beta + 5\,\beta^2). \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Hence $$\bar{\phi}_1(x, y, 0) = \frac{-\omega x}{6\,Mb\,B^4}(x^2 - 6\,B^2\,y^2) \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and $$\phi_1(x, y, 0) = \frac{-\omega x}{6\,Mb\,B^4}(x^2 + 6\,B^4\,y^2). \quad . \quad . \quad . \quad . \quad . \quad (60)$$

In the same way, using the value of ϕ_0 given by (55) it is possible to find an expression for $\bar{\phi}_1$ and hence for ϕ_1, for points in *AFB*. The integration is not completed here but it is easily seen that the value of ϕ_1 obtained will be continuous across the characteristic *AF* with the value given in (60). However, for points on the edge *AB*, where there will in fact be high velocities, the linearised theory breaks down and ϕ_1 becomes infinite owing to the presence in $\partial \bar{\phi}_1/\partial z$ (and $\partial \phi_1/\partial z$) of the term in $\partial^2 \phi_0/\partial z^2$.

3.2. *Symmetrical airscrew*

The second part, $\phi_{(2)}$, of the potential for the flow past the airscrew given by equation (23) is found by considering the flow past the aerofoil given by equation (28):

$$z = y \tan(\Omega x/U) \pm \varepsilon f(x, y).$$

If Ω and ε are assumed small so that ε^2 and Ω^3 (and hence also $\Omega^2 \varepsilon$ and higher powers) may be neglected, the direction cosines of the normal $\mathbf{n}^*$ to the upper surface given by equation (28) are

$$-\left(\frac{\Omega y}{U} + \varepsilon \frac{\partial f}{\partial x}\right), \quad -\left(\frac{\Omega x}{U} + \varepsilon \frac{\partial f}{\partial y}\right), \quad 1 - \frac{\Omega^2}{2U^2}(x^2 + y^2). \quad . \quad . \quad (61)$$

Boundary condition (21) then becomes

$$(\partial \phi/\partial n)_{z = y \tan(\Omega x/U) + 0} = U \varepsilon \partial f/\partial x, \quad . \quad . \quad . \quad (62)$$

and, in the same way, considering the lower surface given by (28), the boundary condition

$$(\partial \phi/\partial n)_{z = y \tan(\Omega x/U) - 0} = -U \varepsilon \partial f/\partial x \quad . \quad . \quad . \quad (63)$$

is obtained. Expanding ϕ as the power series given by (29) and substituting in (62) and (63) gives as the condition on ϕ_0

$$(\partial \phi_0/\partial z)_{z = +0} = -(\partial \phi_0/\partial z)_{z = -0} = U \varepsilon \partial f/\partial x, \quad . \quad . \quad . \quad (64)$$

while the conditions on ϕ_1, ϕ_2 . . . and the equations for ϕ_0, ϕ_1, ϕ_2, . . . are exactly as given in Section 3.1, equations (39), (40) and (31)-(33).

In this case, the boundary condition (64) on $\partial \phi_0/\partial z$ is such that ϕ_0 is an even function of z and so can be found by the use of the method given by Puckett[1] for the symmetrical part of the flow past an aerofoil in a uniform stream. ϕ_1 and the associated $\bar{\phi}_1$ will now be odd functions of z so that the method required is that

given by Ward[3] for the anti-symmetrical part of the flow past an aerofoil in a uniform stream.

3.2.1. *Solution for ϕ_0 and ϕ_1 for an airscrew with symmetrical circular arc cross section*

As an example of the application of the method, an airscrew is considered which is given by equation (23) with $f(x, y)=4(cx-x^2)/c$. This gives a cross section of two equal circular arcs with end points $x=0$ and $x=c$ and maximum thickness $2\varepsilon c$.

The potential ϕ_0 for points independent of the wing tips is simply the well-known potential for steady two-dimensional flow past a circular arc aerofoil and is given by

$$\phi_0=\frac{4U\varepsilon}{Bc}\{(x-Bz)^2-c(x-Bz)\}. \quad . \quad . \quad . \quad (65)$$

With this value for ϕ_0, the boundary condition on the function $\tilde{\phi}_1$ associated with the second approximation ϕ_1 becomes

$$(\partial\tilde{\phi}_1/\partial z)_{z=0}=[4U\varepsilon y/(MBbc)](4x-c). \quad . \quad . \quad (66)$$

Hence, using methods similar to those in Section 3, it is found that

$$\tilde{\phi}_1(x, y, 0)=[4U\varepsilon xy/(MB^2bc)](c-2x), \quad . \quad . \quad (67)$$

and hence that

$$\phi_1(x, y, 0)=[4U\varepsilon xy/(Mbc)](2x-c), \quad . \quad . \quad . \quad (68)$$

for points not affected by the wing tips.

3.3. Summary of results

By superposing the results of Section 3.1 and 3.2, the following solutions are obtained, at points on the part of the surface of the airscrew independent of the tips, for the first and second terms of the series for the potential for the flow past the airscrew surface given by the equation

$$z=y\tan[(\Omega+\omega)x/U]+4\varepsilon(cx-x^2)/c \quad . \quad . \quad . \quad (69)$$

namely,
$$\phi_0(x, y, 0)=-\omega xy/B+4U\varepsilon(x^2-cx)/(Bc) \quad . \quad . \quad . \quad (70)$$

and
$$\phi_1(x, y, 0)=-\frac{\omega x}{6MbB^4}(x^2+6B^4y^2)+\frac{4U\varepsilon xy}{Mbc}(2x-c). \quad . \quad (71)$$

The values of ϕ_0 and ϕ_1 have been found at points on the surface of the airscrew because these are the values required in calculating the forces on the airscrew. The methods of Ward and Puckett do however give the values at points off the surface of the airscrew without difficulty. For example, the expression for ϕ_0 for points which are not affected by the wing tips is (for $z \geqslant 0$)

$$\phi_0(x, y, z) = -\omega y(x/B - z) + \frac{4U\varepsilon}{Bc}[(x - Bz)^2 - c(x - Bz)] . \quad . \quad (72)$$

(On putting $z=0$, this result reduces to equation (70)).

When the results of this section are used in Section 4 for calculating the aerodynamic forces on an airscrew the comparative magnitudes of the two small quantities ε and ω, which have hitherto been quite independent of each other, will have to be taken into consideration. In fact, it will be noticed immediately that ε always appears in these results multiplied by the large velocity U so that it is to be expected, for example that terms of the first order in ε will be larger than those of the first order in ω.

4. Aerodynamic Forces on the Airscrew

The principle of the conservation of momentum can be expressed, in moving axes, in the form

$$\int_S \{ \Delta p\, \mathbf{n} + \rho\, \mathbf{v}(\mathbf{v}' . \mathbf{n}) \}\, dS = 0 \quad . \quad . \quad . \quad . \quad (73)$$

where S is a closed surface fixed relative to the moving axes, Δp is the perturbation pressure and $\mathbf{v}$, $\mathbf{v}'$ are the fluid velocities relative to the fixed and moving axes respectively.

Thus, if S is a closed surface surrounding the aerofoil and sufficiently far from all singularities for the flow to be given accurately by the linearised potential and for $|\nabla\phi|$ to be uniformly small on it, the aerodynamic force $\mathbf{F}$ on the airscrew will be

$$\mathbf{F} = -\int_S \{ \Delta p\, \mathbf{n} + \rho\, \mathbf{v}(\mathbf{v}' . \mathbf{n}) \}\, dS . \quad . \quad . \quad . \quad . \quad (74)$$

With the aid of the continuity equation (4) and the fact that $\rho\mathbf{v}' = \rho_0(\mathbf{U}' + \mathbf{w})$ (equation (13)), equation (74) becomes

$$\mathbf{F} = -\int_S \{ \Delta p\, \mathbf{n} + \rho_0 \operatorname{grad}\phi\, (\mathbf{U}' + \mathbf{w}) . \mathbf{n} \}\, dS \quad . \quad . \quad (75)$$

where Δp and $\mathbf{w}$ are given by equations (10) and (11). By an argument exactly similar to that used by Lighthill[7] for the case of fixed axes, it can be shown that when $\mathbf{F}$ is given by the integral in (75), the surface S can be taken to be any closed surface surrounding the airscrew.

Suppose the airscrew given by equation (23) has a plan form symmetrical about the axis of rotation, and that $f(x, y)$ is an even function of y, as in the particular example given by equation (69). It has been shown in Section 3 that the potential for the flow past such an airscrew can be split into two parts, $\phi_{(1)}$ and $\phi_{(2)}$, and each of these has been expressed as a power series in $\Omega b/a_0$. In finding the forces on the airscrew, the symmetry properties of the various terms are important. It can easily be deduced from the form of the equations and the boundary conditions that $\phi_{0(1)}, \phi_{2(1)}, \ldots$ and $\phi_{1(2)}, \phi_{3(2)}, \ldots$ are odd functions of z while $\phi_{1(1)}, \phi_{3(1)}, \ldots$ and $\phi_{0(2)}, \phi_{2(2)}, \ldots$ are even functions of z. Also, for the particular type of plan form and cross section chosen, $\phi_{0(1)}, \phi_{2(1)}, \ldots$ and $\phi_{1(2)}, \phi_{3(2)}, \ldots$ are odd functions of y and $\phi_{1(1)}, \phi_{3(1)}, \ldots$ and $\phi_{0(2)}, \phi_{2(2)}, \ldots$ are even functions of y.

If now (75) is integrated over the upper and lower surfaces of the airscrew and use is made of these symmetry properties, the following expressions are obtained for the torque and the drag

$$Q = -4\rho_0 \int\int y \left[U\,\phi_{0x(1)} + \frac{\Omega b}{a_0} \left\{ U\,\phi_{1x(2)} + \frac{y\,a_0}{b}\,\phi_{0x(2)} + \frac{U\,xy}{Mb}\,\phi_{0xz(2)} \right\} \right]_{z=+0} dS + \\ + O(\rho_0\,\Omega^2\,\omega\,b^5\,c/a_0) \qquad (76)$$

in the direction of the angular velocity and

$$D = -4\,\rho_0 \int\int \left[\phi_{0x(1)}\,\phi_{0z(1)} + \phi_{0x(2)}\,\phi_{0z(2)} + \Omega y\,\phi_{0x(1)} + \frac{\Omega^2 b^2}{a_0^2} \left\{ \frac{U\,xy^2}{M^2 b^2}\,\phi_{0xz(2)} + \right. \right. \\ \left. \left. + \frac{a_0\,y}{b}\,\phi_{1x(2)} + \frac{a_0\,y^2}{M\,b^2}\,\phi_{0z(2)} \right\} \right]_{z=+0} dS + O(\rho_0\,\varepsilon\,\omega\,\Omega\,b^3\,c). \qquad (77)$$

In each case the integration is over the half of the airscrew plan form for which $y \geqslant 0$.

Equations (76) and (77) show that the results† obtained from the quasi-static theory for the torque and the drag, namely,

$$Q = -4\,\rho_0\,U \int\int y\,[\phi_{0x(1)}]_{z=+0}\,dS \qquad (78)$$

and

$$D = -4\,\rho_0 \int\int [\phi_{0x(1)}\,\phi_{0z(1)} + \phi_{0x(2)}\,\phi_{0z(2)}]_{z=+0}\,dS \qquad (79)$$

†Compare the result given by Ward[3] (equation (60)) for the drag on an aerofoil.

are the first approximations obtained by retaining only the terms independent of Ω. However, it will be seen when numerical values are inserted that, depending on the relative magnitudes of the physical quantities, these terms are not necessarily the largest ones.

4.1. *Application to special type of airscrew*

These results for the torque and the drag are now applied to the particular airscrew with surfaces given by equation (69) and with plan form which is rectangular but which has its corners cut off along the characteristics through the tips of the leading edge, *i.e.* the plan form is *AFED* in Fig. 1. By substituting in equations (76) and (77) the values obtained for $\phi_{0(1)}$, $\phi_{0(2)}$, $\phi_{1(2)}$ from equations (70)-(72), the following results are derived for the torque and the drag.

$$Q=\frac{1}{3}\rho_0\omega U\left[b^4-\left(b-\frac{c}{B}\right)^4\right]+\frac{4}{3}\rho_0 U\Omega\varepsilon(B+1)\left[b^4-\left(b-\frac{c}{B}\right)^4\right]-$$
$$-\frac{8\rho_0 U\Omega\varepsilon B(B+1)}{15c}\left[b^5-\left(b-\frac{c}{B}\right)^4\left(b+\frac{4c}{B}\right)\right]+O(\rho_0\Omega^2\omega b^5 c/a_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

and

$$D=\frac{1}{3}\rho_0\omega(\Omega+\omega)\left[b^4-\left(b-\frac{c}{B}\right)^4\right]+\frac{32\rho_0 U^2\varepsilon^2 c}{3B}(2b-c/B)+O(\rho_0\varepsilon\omega\Omega b^3 c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Consider the special case in which the main stream has Mach number $M=\sqrt{2}$ so that $B=1$ and the velocity $U=a_0\sqrt{2}$ where, as an example, the speed of sound, a_0, and the density ρ_0 are assumed to have the values given by Pankhurst and Conn[8] for conditions in the standard atmosphere at a height of 20,000 ft.

If λ, μ, χ and ζ are non-dimensional parameters defined by the relations

$$\omega=\lambda\Omega,\quad b=\mu c,\quad \chi=\Omega b/a_0,\quad \Omega Q=P=\zeta\pi c^2\rho_0 U^3, \quad . \quad . \quad (82)$$

then the expressions for the power produced by the windmill and the power absorbed by the drag become

$$P=\Omega Q=(\sqrt{2}/3)\rho_0 a_0\Omega^2 c^4\times 10^{-7}\times\left\{[\mu^4-(\mu-1)^4](\lambda+8\varepsilon)-\right.$$
$$\left.-\frac{16\varepsilon}{5}[\mu^5-(\mu-1)^4(\mu+4)]+O\left(\frac{3\lambda\mu^3\chi}{\sqrt{2}}\right)\right\}\text{ watts}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

and
$$UD=(\sqrt{2}/3)\rho_0 a_0\Omega^2 c^4\times 10^{-7}\times\left\{[\mu^4-(\mu-1)^4]\lambda(\lambda+1)+\right.$$
$$\left.+\frac{64\varepsilon^2\mu^2}{\chi^2}(2\mu-1)+O(3\varepsilon\lambda\mu^4)\right\}\text{ watts}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

From (83) and (84) it can be seen that the efficiency, $\eta = \Omega Q/(U D)$, depends only on the four parameters λ, μ, ε, χ. For given values of μ and χ, the efficiency takes the form

$$\eta = \frac{\lambda + \kappa \varepsilon}{\lambda^2 + \lambda + \nu \varepsilon^2} \quad . \quad . \quad . \quad . \quad . \quad (85)$$

where κ and ν depend only on μ and χ. The way in which η varies with λ and ε depends on the magnitudes of κ and ν and it may be seen that there are three main types of behaviour possible. However, with the range of values of κ and ν required by the approximations of the theory, only two of these types occur. In each of these, η is bounded for positive values of λ and reaches a maximum value, η_m, for a certain value, λ_m, of λ, with corresponding value, ζ_m, of the "power coefficient" ζ. λ_m is given by the equation

$$\lambda_m = \{ (\kappa^2 + \nu) \varepsilon^2 - \kappa \varepsilon \}^{\frac{1}{2}} - \kappa \varepsilon \, . \quad . \quad . \quad . \quad . \quad (86)$$

It is clearly desirable to operate the windmill at the highest possible efficiency; therefore, keeping Ωc fixed, the variation with μ and ε of λ_m is considered and the corresponding values of μ_m and ζ_m. This variation is shown in Figs. 3, 4 and 5, for values of μ between 2.0 and 10.0 and of ε between 0.01 and 0.05, keeping $\Omega c = 1{,}000$ cm./sec. Fig. 5 also indicates the actual power obtained from a windmill of chord 2 cm. under these conditions. From the results shown in these three diagrams, it is seen that the best working conditions for given Ωc are obtained by taking ε as small and μ as large as possible. This means taking the thickness of the airscrew as small and the span as large as possible for given chord and, in each case, the limit will be imposed by the strength of the material of the airscrew. These conditions not only give the highest value for η_m but also ensure that λ_m has the lowest value possible. This is desirable as the linearised theory has been developed on the assumption that ω and hence λ is small.

To obtain an indication of the limits to μ and ε, an estimate is made of the maximum stress in the blade and this is compared with the elastic limit for the material of the airscrew. The bending moment due to the pressure forces on the blade is a maximum at the root of the blade and is equal there to half the total torque on the airscrew. The greatest thickness of the blade at the root is $2\varepsilon c$ and hence, by the well-known formula relating the stress in a beam to the bending moment (*see*, for example, Lamb[9]), the maximum stress, S', is given roughly by

$$S' = \varepsilon c Q/(2I) \quad . \quad . \quad . \quad . \quad . \quad (87)$$

where $\frac{1}{2}Q$ = bending moment at root of blade

and I = quadratic moment of the root section with respect to its neutral line.

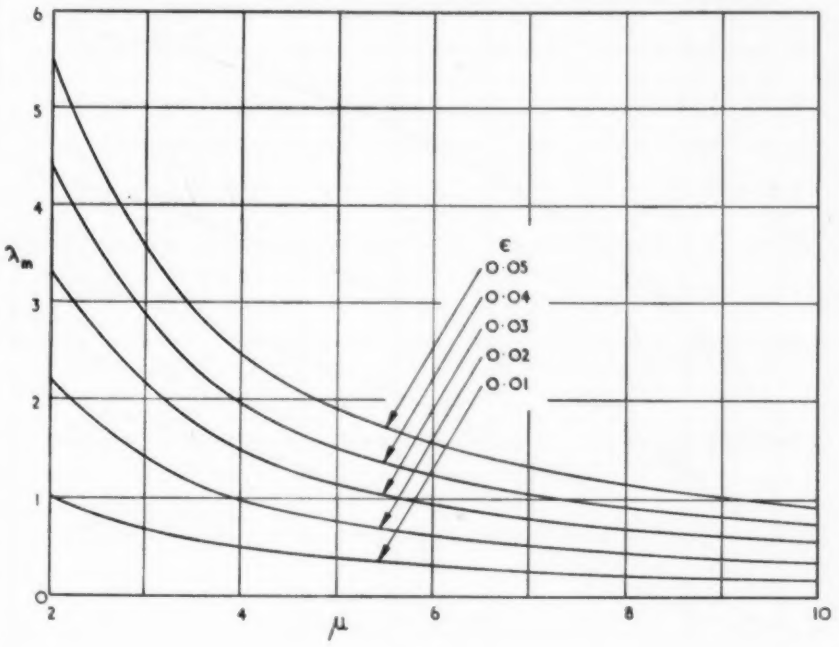

Fig. 3.

Variation of λ_m with μ and ϵ for $\Omega c = 1{,}000$ cm./sec.

The maximum stress must be less than, say, half the elastic limit of the material of the airscrew. In Fig. 6 are shown the values of the ratio of these two stresses for different values of μ and ε when the airscrew is working at maximum efficiency and $\Omega c = 1{,}000$ cm./sec. It is assumed that the airscrew is made of spring steel with an elastic limit of 70,000 lb./in.^{2} From these results, the values of μ and ε can be obtained for which the maximum stress is less than half the elastic limit (taking a safety factor of 2) and it is seen, for example, that μ must not be greater than 3.0 when $\varepsilon = 0.03$. In practice it would be necessary to consider sets of results of this type for different values of the angular velocity.

The variations in λ_m, η_m and ζ_m with χ, keeping μ and ε constant are now considered. In Fig. 7, the values of these parameters are given for values of χ between 0 and 0.76, keeping $\mu = 3$ and $\varepsilon = 0.03$, as a particular example. For values of χ greater than approximately 0.93 the variation of η with λ changes over to the type in which η decreases steadily from infinity to zero as λ goes from zero to infinity. This is a non-physical solution and is ruled out because these values of χ are evidently too large for the linearised theory to be valid. The variation in ζ_m

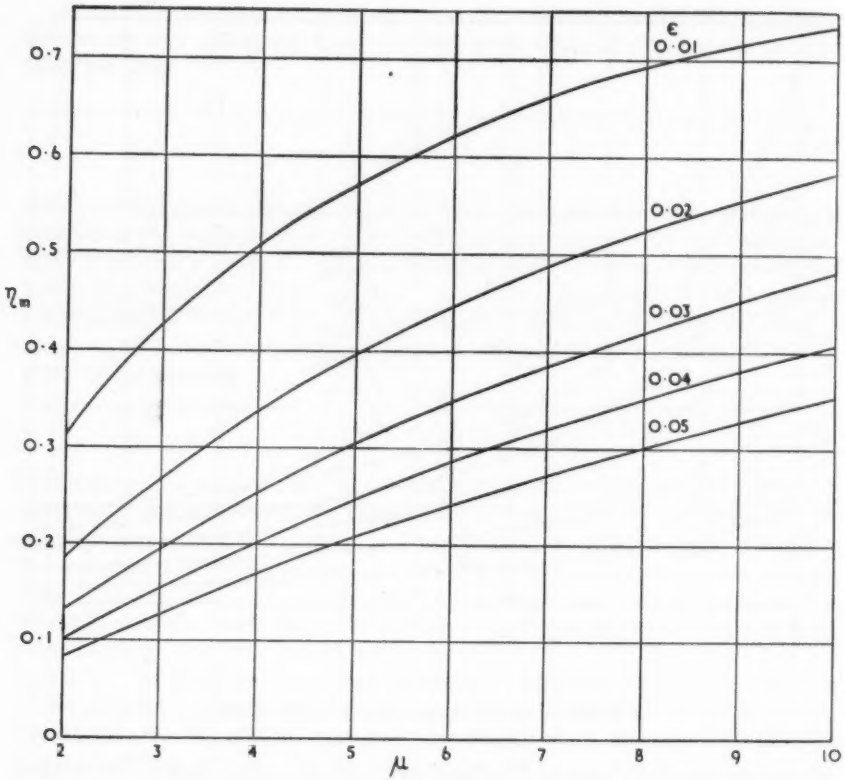

Fig. 4.

Variation of η_m with μ and ϵ for $\Omega c = 1{,}000$ cm./sec.

with χ also shows how the magnitude of χ must be limited. The initial steady increase of ζ_m with χ is followed by a rapid decrease after the maximum is reached. It can be seen from the algebraic dependence of λ_m and ζ_m on χ that they each become zero for values of χ approximately equal to 0.93 while, at the same value of χ, η is approximately unity. This decrease in power with increase of tip speed is not to be expected on physical grounds and indicates that the linearised theory is not valid for such large values of χ. That this is so is further indicated by the fact that there is no significant change in the shape of the airscrew at or near the value $\chi = 0.93$. The equation (19) of the basic airscrew surface can be written

$$z = y \tan [\chi x/(\mu Mc)] \quad . \quad . \quad . \quad . \quad (88)$$

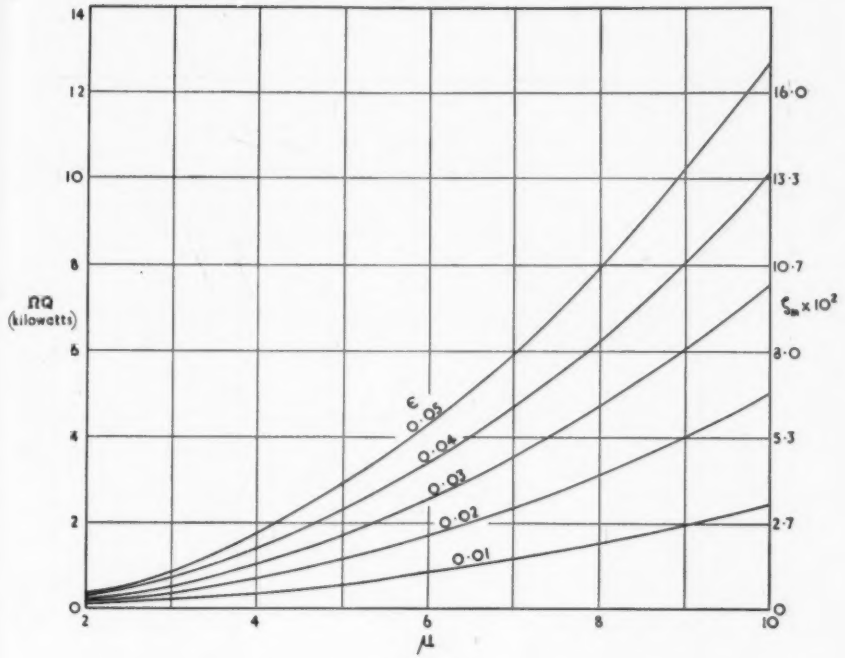

Fig. 5.
Variation of ζ_m and power ΩQ (for $c=2$ cm.) with μ and ϵ for $\Omega c=1{,}000$ cm./sec.

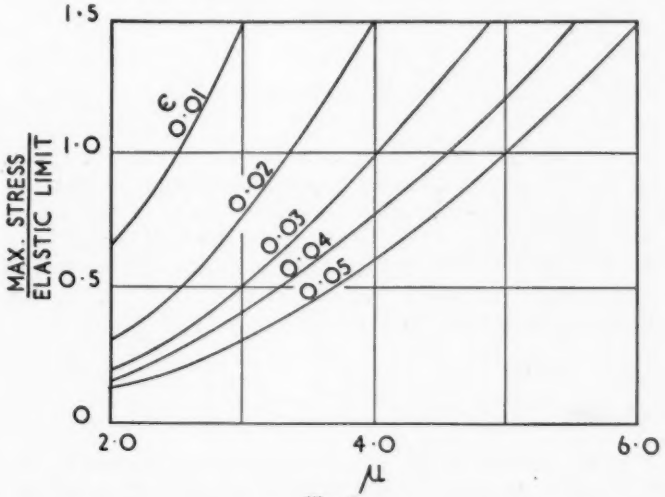

Fig. 6.
Variation with μ and ϵ of the ratio of the maximum stress in the airscrew blade to the elastic limit of steel.

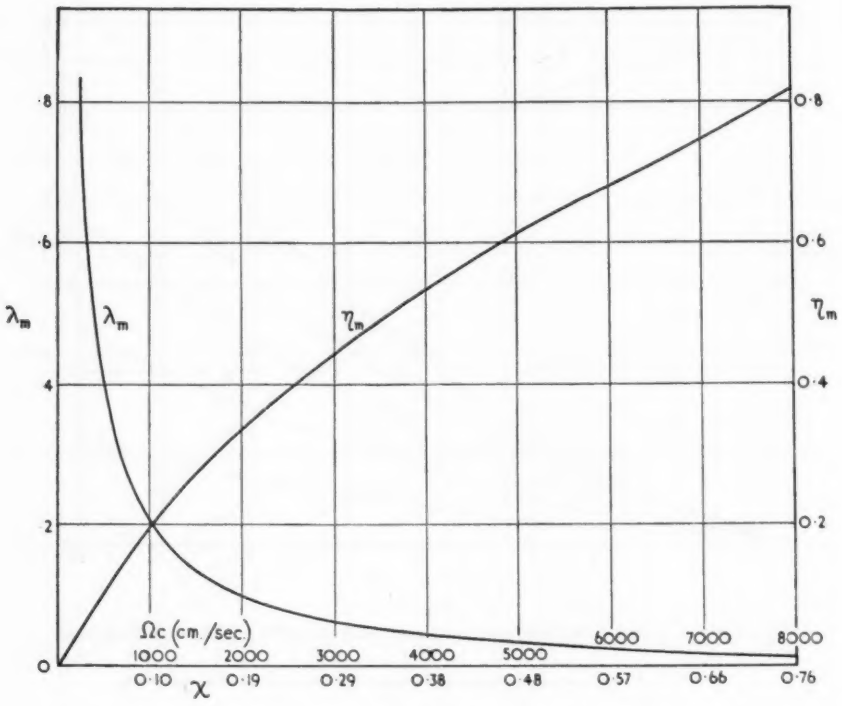

Fig. 7(*a*).

Variation of λ_m and η_m with χ for $\mu=3$, $\epsilon=0.03$.

which, for $\mu=3$, $M=\sqrt{2}$ and $\chi=0.93$, becomes approximately $z=y\tan(13\,x^o/c)$. It is evident that this surface is twisted only slightly from the plane $z=0$ and does not change in any important way when χ changes slightly. In the case considered, it seems reasonable to suppose that the theory remains valid for values of χ up to 0.5. As far as variations of λ_m and η_m with χ are concerned, the best results are obtained for high values of χ which produce high values of η_m and low values of λ_m, the merits of which have been discussed previously.

5. Conclusion

Apart from the estimate of the limits imposed by the strength of the material of the airscrew on the aspect and thickness ratios of the blades, no attempt has been made here to effect the necessary compromise between theoretical and practical requirements in designing a supersonic windmill. An indication of the procedure

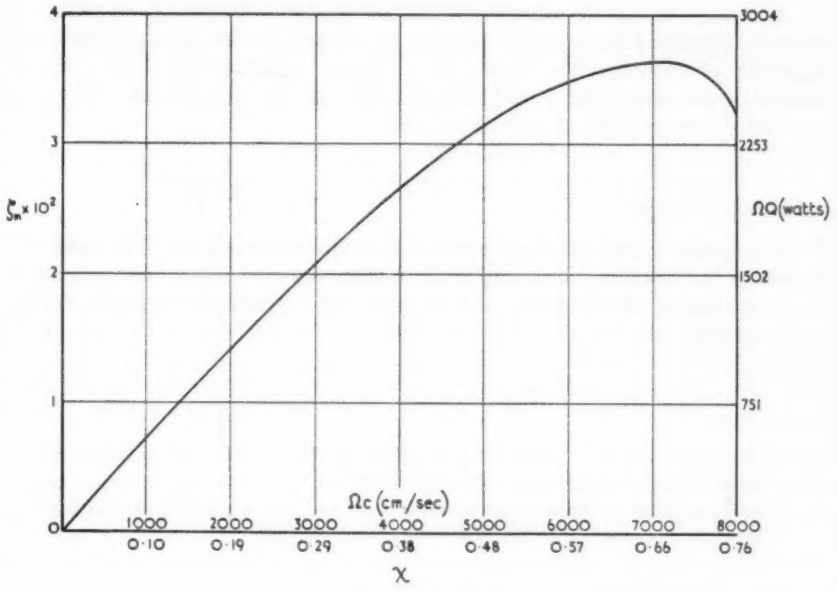

Fig. 7(*b*).

Variation of ζ_m and power ΩQ (for $c=2$ cm.) with χ for $\mu=3$ and $\epsilon=0{\cdot}03$.

can be obtained from the results which have been given in Section 4. To take a specific case, it may be seen from Fig. 6 that the values $\mu=3$, $\varepsilon=0.03$ appear to give a practicable shape for a steel airscrew of the type considered, working at maximum efficiency and rotating so that $\Omega c=1{,}000$ cm./sec. (and $\chi=0.1$ approximately). Reference to Fig. 4 shows that, under these conditions, this particular windmill has an efficiency of about 20 per cent. while Fig. 5 gives the power obtained from such a windmill with a chord of 2 cm. as approximately 0.54 kilowatts. Now Fig. 7 shows that the maximum efficiency of an airscrew and the power obtained from it increase as the angular velocity increases. This suggests that, although the maximum stresses will also be increased, higher efficiencies can be obtained by increasing the angular velocity. In any particular case, it would clearly be desirable to work out all these results for a range of values of Ωc. The method used for integrating the equations is a very general one so that the theory can be applied to a wide range of airscrews and, in particular, the straight leading edge and constant cross section chosen for simplicity in the example worked out in this paper are by no means essential. Accounts of the methods used in propeller design which presents problems similar to those in the case considered in this paper may be found in such works as those by Theodorsen[10] and Bairstow[11].

In cases where greater accuracy is required than was obtained in the particular example considered, it is always possible to calculate further terms of the series expansion for the potential in powers of χ. Terms in higher powers of χ can then be retained in calculating the torque and the drag and the efficiency and the power obtained to any required degree of accuracy.

Acknowledgments

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Pohlhausen's Method for Three-Dimensional Laminar Boundary Layers

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(*Summary*: Wild has extended to three dimensions the von Kármán-Pohlhausen method for the laminar boundary layer flow over a fixed obstacle and used the method for an infinite yawed elliptic cylinder in a stream. In this paper the method is tested in two ways (which may be called full-Pohlhausen and semi-Pohlhausen) for the case of an infinite yawed cylinder when the velocity outside the boundary layer over the surface normal to the generators is of the form $U=cx^m$. The exact solution is known in this case.

A table of the skin friction, displacement thickness and momentum thickness is given for various values of β $[=2m/(m+1)]$, and the agreement is found to be fairly good for $\beta>0$ (accelerated flow) but not so good for $\beta<0$ (retarded flow).)

1. Introduction

Wild[1] has given a method for calculating the three-dimensional laminar boundary layer over an infinite yawed elliptic cylinder by an extension of the method of Pohlhausen and von Kármán. While this method is satisfactory under a favourable pressure gradient, it is not so good when the flow is retarded. Nevertheless it gives useful results and so the extension of the method to yawed infinite cylinders is worth consideration. This paper considers yawed infinite cylinders when the potential flow normal to the generators is of the form $U=cx^m$. The exact solutions for this case are known[2] and so it is possible to check the method. The results are fair for positive values of m, less good for negative values. Even in the simple case given by Wild the numerical work is fairly heavy, although straightforward. It would seem that it will be very heavy indeed if the method is extended to more general cases.

Notation

x, y, z rectangular Cartesian co-ordinates, the x-axis being tangential to the surface and normal to the generators, the y-axis being parallel with the generators

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u, v, w	velocity components in the boundary layer in the directions of x, y and z respectively
v', v''	first and second derivatives of v with respect to z
U, V	main stream velocity components in the directions of x and y respectively
U', U''	the first and second derivations of U with respect to x
c, m	constants in the equation for the velocity $U = cx^m$
β	$2m/(m+1)$
ρ	density
p	pressure
μ	coefficient of viscosity
ν	kinematic viscosity $= \mu/\rho$
δ_y	boundary layer thickness of the velocity component v
δ_x	boundary layer thickness of the velocity component u
λ	$U'\delta_x^2/\nu$
$f(\lambda), g(\lambda)$	functions defined by equation (7)
δ_x^*	displacement thickness of boundary layer for velocity component u $\left(= \int_0^{\delta_x} (1-u/U)\,dz\right)$
δ_y^*	displacement thickness of boundary layer for velocity component v $\left(= \int_0^{\delta_y} (1-v/V)\,dz\right)$
τ_x, τ_y	components of skin friction in directions of x and y respectively
K	δ_y/δ_x
k^2	$\frac{1}{2}(m+1)$
T_x	$[\tau_x/(\rho U^2)]\sqrt{[\pm Ux/(\nu m)]}$
T_y	$[\tau_y/(V\mu k)]\sqrt{(\nu x/U)}$
θ_y	momentum thickness of boundary layer for velocity component v $\left(= \int_0^{\delta_y} \frac{v}{V}\left(1-\frac{v}{V}\right)dz\right)$
Θ_y	$\theta_y k\sqrt{[U/(\nu x)]}$
D_x^*	$\delta_x^*\sqrt{[\pm mU/(\nu x)]}$
D_y^*	$\delta_y^* k\sqrt{[U/(\nu x)]}$

D $k\sqrt{[U/(\nu x)]}\,\delta_y$

Y $k\sqrt{[U/\nu x)]}\,z$

ξ Y/D

I integral defined immediately after equation (10)

F a function of K and λ defined by the relation $\frac{d}{dx}(FU\delta_x) = -2\nu/(K\delta_x)$

2. The Equations of Motion

Taking the x-axis tangential to the surface and normal to the generators, the y-axis parallel to them and the z-axis perpendicular to those of x and y, the Prandtl equations of boundary layer flow become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2},$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2},$$

$$0 = \partial p/\partial z,$$

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0.$$

The main stream velocity components are taken as U and V, where V is constant.

In the case of infinite yawed cylinders none of the functions changes as y changes, and we have

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$0 = \partial p/\partial z, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\partial u/\partial x + \partial w/\partial z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(1), (3) and (4) are the ordinary boundary layer equations for two-dimensional flow in the xz-plane, of which there are many methods of solution including Pohlhausen's. (2) is a new equation for v, the cross flow in the boundary layer.

By integration of (2) with respect to z, or directly from momentum considerations (*see* also Tetervin[3]) we obtain as a momentum equation

$$\frac{d}{dx}\int_0^\infty u(v-V)\,dz = -\nu\left(\frac{\partial v}{\partial z}\right)_{z=0} \quad . \quad . \quad . \quad . \quad (5)$$

The von Kármán-Pohlhausen assumptions are (*a*) that this equation holds over a small boundary layer thickness δ_y, where here δ_y is a function of x only, (*b*) the velocity v may be expanded in powers of z/δ_y up to the fourth power. When the usual boundary conditions $v=v''=0$ when $z=0$; $v'=v''=0$, $v=V$ when $z=\delta_y$ are applied (the quartic will not of course fulfil all the boundary conditions) we obtain

$$v/V = 2(z/\delta_y) - 2(z/\delta_y)^3 + (z/\delta_y)^4$$

as given by Wild.

The two-dimensional Pohlhausen method is too well known to be described in detail here. The best account is given by Howarth[4] who repeated it in a shortened form in Ref. 5. Suffice it to say that u/U is expanded as a quartic in z/δ_x; the boundary conditions lead to the equation

$$\frac{u}{U} = 2\left(\frac{z}{\delta_x}\right) - 2\left(\frac{z}{\delta_x}\right)^3 + \left(\frac{z}{\delta_x}\right)^4 + \frac{\lambda}{6}\left(\frac{z}{\delta_x}\right)\left(1-\frac{z}{\delta_x}\right)^3, \quad . \quad . \quad (6)$$

where $\lambda = U'\delta_x^2/\nu$.

Substituting in the Kármán momentum equation leads to

$$d\lambda/dx = (U'/U)f(\lambda) + (U''/U')g(\lambda), \quad . \quad . \quad . \quad (7)$$

dashes denoting differentiation with respect to x. $f(\lambda)$ and $g(\lambda)$ are known rational functions of λ.

Here the velocity external to the layer is considered as $U=cx^m$, which makes (7) become

$$d\lambda/dx = (1/x)\{mf(\lambda) + (m-1)g(\lambda)\}.$$

This equation can be integrated directly and it appears that it has no solution except $\lambda=$constant; this is determined by solving the equation

$$mf(\lambda) + (m-1)g(\lambda) = 0. \quad . \quad . \quad . \quad . \quad (8)$$

That λ should be a constant is expected by similarity considerations; the

solution is exactly known (Falkner and Skan[6][7] and Hartree[8]) and the velocity profiles are known to be "similar."

Equation (8) is a cubic and the correct root is determined by physical considerations in more general cases. Dryden[9] gives a curve of λ against β, where $\beta=2m/(m+1)$.

It is worth while showing the accuracy of this solution before proceeding to the three-dimensional case, and the results for the skin friction τ_x and displacement thickness δ_x^* are given in Table I for selected values of β. In this table

$$T_x=[\tau_x/(\rho U^2)]\sqrt{[\pm Ux/(\nu m)]}, \quad D_x^*=\delta_x^*\sqrt{[\pm mU/(\nu x)]}.$$

It can be seen that the accuracy is reasonable except for the skin friction in the case of negative β. It is well known that this method gives inaccurate results in a retarded region.

3. The Equation for v

We have

$$\frac{d}{dx}\int_0^{\delta_y} u(v-V)\,dz=-\nu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

where v is given by the equation

$$v/V=2(z/\delta_y)-2(z/\delta_y)^3+(z/\delta_y)^4 \quad . \quad . \quad . \quad . \quad (9)$$

for $0\leqslant z\leqslant\delta_y$ together with $v=V$ for $z\geqslant\delta_y$.

Also u/U is given by the quartic (6) in z/δ_x, if $0\leqslant z\leqslant\delta_x$ together with $u=U$ if $z\geqslant\delta_x$.

TABLE I

β	λ	T_x (*approx.*)	T_x (*exact*)	D_x^* (*approx.*)	D_x^* (*exact*)
−0.1988	−6.869	0.326	0.000	0.936	1.052
−0.1	−2.170	0.886	1.010	0.469	0.446
0.2	2.509	1.527	1.536	0.442	0.440
0.6	5.268	1.254	1.286	0.588	0.592
1.0	7.052	1.196	1.233	0.641	0.648
1.6	9.349	1.164	1.202	0.679	0.688
2.0	11.984	1.158	1.193	0.693	0.703

The integral will be different according as δ_x is greater or less than δ_y owing to the differences in the forms of u/U and v/V inside and outside the boundary layer. u/U and v/V do not fulfil all the boundary conditions.

Writing $\delta_y = K\delta_x$ we obtain after some reduction

$$\frac{d}{dx}(FU\delta_x) = -\frac{2\nu}{K\delta_x},$$

where F is a function of K and λ.

This function has the value

$$F_1 = 0.3 - 0.00833\lambda + (1/K)(-0.13333 + 0.00556\lambda) + (1/K^3)(0.02143 - 0.00119\lambda) + \\ + (1/K^4)(-0.00556 + 0.00033\lambda) - 0.3K$$

if $K > 1$, and the value

$$F_2 = -K^2(0.13333 + 0.01111\lambda) + K^3(0.01190\lambda) + K^4(0.02143 - 0.00536\lambda) + \\ + K^5(-0.00556 + 0.00093\lambda)$$

if $K < 1$.

These results agree with those of Wild, although written differently.

Using the equations $\lambda = U'\delta_x^2/\nu = \text{constant}$, and $U = cx^m$ we obtain

$$dK/dx = -(2/K + \lambda F/\beta)\Big/(x\,dF/dK).$$

At the forward stagnation line $x = 0$ and if $dK/dx = \infty$ is excluded as physically impossible, the numerator must also vanish for $x = 0$. The equation has in fact a singular point at $x = 0$, and in all the instances tried it is found that the only solution of the equation is $K = \text{constant}$, the constant being determined from the equation

$$2\beta + \lambda KF = 0.$$

In the case $K > 1$ this is a quintic for K and in the case $K < 1$ it is a sextic. In each of the cases tried there is only one physically acceptable root in the desired range.

4. Results

The results have been worked out for the values -0.1988, -0.1, 0.2, 1.0, 1.6, and 2.0 for β, and are given in Table II. To obtain results for the skin friction,

$$\tau_y = \mu(\partial v/\partial z)_{z=0} = 2V\mu/\delta_y,$$

and writing

$$T_y=[\tau_y/(V\mu k)]\sqrt{(\nu x/U)}=(2/K)\sqrt{(\beta/\lambda)},\quad [k^2=\tfrac{1}{2}(m+1)]$$

a direct comparison is possible with $G'(0)$ in Ref. 2.

For the displacement thickness

$$\delta_y^*=\int_0^{\delta_y}(1-v/V)\,dz=0.3\,\delta_y,$$

writing $$D_y^*=\delta_y^*\,k\sqrt{[U/(\nu x)]}=0.3\,K\sqrt{(\lambda/\beta)},$$

direct comparison can be made with $\int_0^{\infty}(1-G)\,dY$ in Ref. 2.

For the momentum thickness

$$\theta_y=\int_0^{\delta_y}\frac{v}{V}\left(1-\frac{v}{V}\right)dz=0.11746\,\delta_y,$$

and writing $$\Theta_y=\theta_y\,k\sqrt{[U/(\nu x)]}=0.11746\,K\sqrt{(\lambda/\beta)},$$

a direct comparison can be made with $\int_0^{\infty}G\,(1-G)\,dY$ in Ref. 2.

TABLE II

β	λ	K	T_y (*approx.*)	T_y (*exact*)	D_y^* (*approx.*)	D_y^* (*exact*)	Θ_y (*approx.*)	Θ_y (*exact*)
−0.1988	−6.869	0.8574	0.3968	0.3258	1.512	1.6696	0.5920	0.5937
−0.1988	(*Alternative method*)		0.360	0.3258	1.667	1.6696	0.653	0.5937
−0.1	−2.170	0.9412	0.4561	0.4332	1.316	1.3043	0.5160	0.5151
−0.1	(*Alternative method*)		0.460	0.4332	1.304	1.3043	0.511	0.5151
0.2	2.509	1.089	0.5188	0.5069	1.1565	1.1376	0.4497	0.4528
0.6	5.268	1.2154	0.5555	0.5467	1.0801	1.0646	0.4173	0.4229
1.0	7.052	1.3043	0.5773	0.5704	1.0393	1.0265	0.4044	0.4069
1.6	9.349	1.3819	0.5985	0.5938	1.0025	0.9917	0.3923	0.3925
2.0	11.984	1.3400	0.6099	0.6053	0.9828	0.9759	0.3872	0.3852

It is seen that the agreement is quite good for positive β, that is for accelerated flow in the x direction. Even in the case of retarded flow the agreement is better than in the two-dimensional case. As it depends on the two-dimensional solution, which is so inaccurate here, it cannot be expected to be good.

5. Alternative Procedure

In view of the foregoing it might seem better to use the known two-dimensional solution, and to use Pohlhausen's method for the cross flow only.

In the equation

$$\frac{d}{dx}\int_0^{\delta_y} u\,(v-V)\,dz = -\nu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

the Pohlhausen quartic (9) is substituted for v/V and the Hartree value $UF'(Y)$ for u, where $Y = k\sqrt{[U/(\nu x)]}\,z$, $k^2 = \frac{1}{2}(m+1)$ (the notation is that of Goldstein[5]).

On putting $D = k\sqrt{[U/(\nu x)]}\,\delta_y$, $\xi = Y/D$, and assuming in virtue of the known "similarity" that D is constant we obtain after some reduction

$$ID^2 = 2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where

$$\mathrm{I} = \int_0^1 (-2\xi + 2\xi^3 - \xi^4 + 1)F'(D\xi)\,d\xi.$$

To solve (10) D is given a set of suitable values, I is evaluated numerically using Hartree's table, and by interpolation the value which satisfies equation (10) is found.

In this paper the method has only been used approximately for the values -0.1 and -0.1988 for β. It is found that

$$T_y = 2/D, \quad D_y^* = 0.3\,D, \quad \Theta_y = 0.11746\,D. \quad . \quad . \quad . \quad (11)$$

The values are given in Table II. It will be seen that while some of the values are improved others are not so good as before. As can be seen from equations (11), the three quantities tabulated depend on each other and it is impossible to make them all agree at the same time. It shows in fact that the fundamental assumption of a quartic is not quite good enough in this region.

If the velocity profiles are plotted it is found that the curves for the two methods (exact and approximate) run very close together for all the values of β tried here except $\beta = -0.1988$. In Fig. 1 the curves show the exact solution, the first method

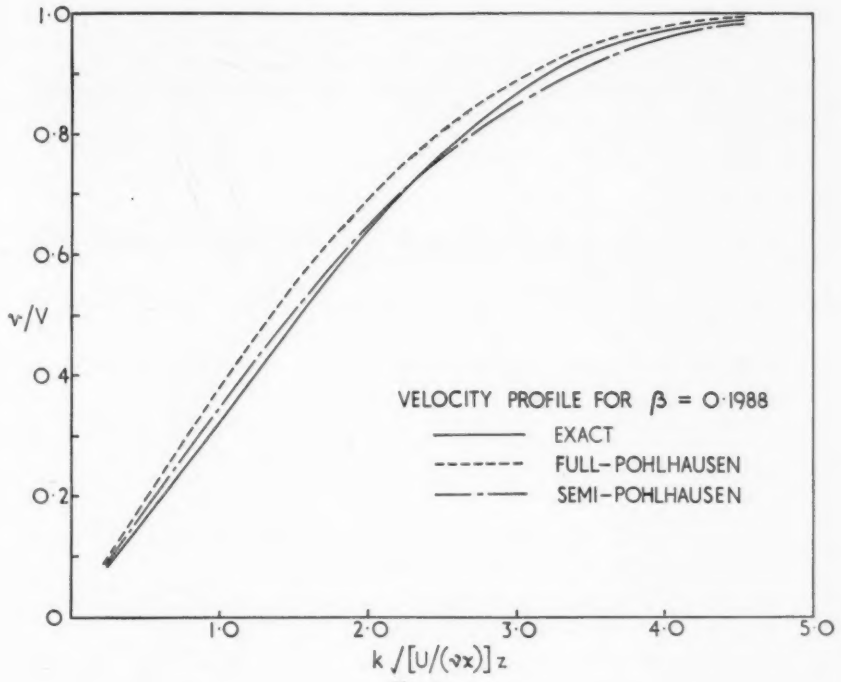

Fig. 1.

given here (full-Pohlhausen) and the alternative procedure (semi-Pohlhausen) all for $\beta = -0.1988$. These curves show clearly that the assumed quartic velocity profile is not sufficiently good in the region of retarded flow in the main stream.

6. Conclusions

It seems doubtful if the accuracy of the method would be worth the labour involved in more general cases. It is possible theoretically, using the momentum equations of Tetervin[3], but the labour would seem to be prohibitive. Prandtl[10] has suggested a method, but these methods would require testing by knowledge of more "exact" solutions than are at present known. Apart from flows involving axial symmetry, and from yawed infinite cylinders, there appears to be only one exact three-dimensional solution known. (Carrier[11].)

Taylor[12] and Binnie and Harris[13] have used Pohlhausen's method for the solution of swirl problems but it may be remarked that they assume that lengths equivalent to what are called δ_x and δ_y in the present paper are to be taken as equal.

It is certainly true that in the exact solution given in Ref. 2 the boundary layer thicknesses in the two directions (reckoned on any consistent basis, *e.g.* the usual 99 per cent. of the main stream velocities) are considerably different. Their ratio in fact agrees fairly well with K of this paper. It is hoped to discuss this matter in a later paper.

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Supersonic Flow Past Bodies of Revolution with Thin Wings of Small Aspect Ratio

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(*Summary*: The method developed by G. N. Ward[1] for the treatment of slender pointed bodies in a uniform supersonic stream is applied to three special cases.

(*i*) Supersonic flow past a body of revolution with thin wings of symmetrical section and of small aspect ratio at zero incidence.

(*ii*) Supersonic flow past a body of revolution with plane wings of small aspect ratio set at incidence to the body, the whole being at incidence to the stream.

(*iii*) Supersonic flow past a body of revolution with a plane fin of small aspect ratio set at incidence, the whole being at incidence to the stream.

The pressure distribution on the wing has been calculated for a special case of (*i*) and is given in the Appendix.)

1. Introduction

In Ref. 1 the well-known linearised equation for the supersonic isentropic flow of a perfect inviscid compressible fluid past slender pointed bodies of thickness ratio t (t being small compared with unity) is solved by operational methods. It is shown that if the perturbation potential is $U\phi$, the function ϕ may be approximated to within a factor $1+O(t^2 \log t)$ by the function ϕ_0 obtained as follows. Taking any section of the body by a plane perpendicular to the undisturbed stream, a solution ϕ_0 of the two-dimensional Laplace equation $\nabla^2 \phi_0 = 0$ is determined, which satisfies certain boundary conditions on the boundaries of this body section. The boundary conditions which must be applied are such that the solution is found to be identical with the solution of a problem in two-dimensional incompressible flow. If the plane of section is moved downstream with velocity U, the shape and position of the body section relative to the plane will change; this may be regarded as the motion of a cylinder of varying cross-section in an incompressible fluid at rest at infinity. The potential ϕ_0 at any point is equal to the potential at that point, in the plane of section through the point, due to the incompressible motion just described.

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A system of right-handed orthogonal axes Ox, Oy and Os is chosen with the origin at the nose of the body, the axis of s being parallel to the undisturbed stream and extending downstream; the scale of s is chosen so as to make the body of unit length. Let $S(s)$ be the area of the section of the body by a plane $s=$constant, let $z=x+iy$ and let $w_0=\phi_0+i\psi_0$, where ψ_0 is the harmonic conjugate of ϕ_0. For sufficiently large $|z|$, w_0 may be expanded in the form

$$w_0=\phi_0+i\psi_0=a_0\log z+b_0+\sum_{N=1}^{\infty}a_N z^{-N},$$

where a_0, b_0 and a_N are functions of s. Let $x_g(s)$ and $y_g(s)$ be the co-ordinates of the centre of area of the body section in the plane $s=$constant. Let ρ_1 be the density of the fluid in the undisturbed stream and $z_g=x_g+iy_g$. It is shown in Ref. 1 that if X and Y are the components of total force on the body in the directions of x and y increasing respectively, then

$$F/(\tfrac{1}{2}\rho_1 U^2)=(X+iY)/(\tfrac{1}{2}\rho_1 U^2)=4\pi(a_1)_{s=1}+2S'(1)z_g(1)+2S(1)z_g'(1), \qquad (1)$$

where primes denote differentiation with respect to s.

The drag force D in the direction of s increasing is given by

$$\frac{D}{\tfrac{1}{2}\rho_1 U^2}=\frac{1}{2\pi}\int_0^1\int_0^1\log\frac{1}{|s-\sigma|}S''(s)S''(\sigma)\,d\sigma\,ds-\frac{S'(1)}{2\pi}\int_0^1\log\frac{1}{|s-\sigma|}S''(\sigma)\,d\sigma-$$

$$-\left(\int_\Gamma\phi_0\frac{\partial\phi_0}{\partial\nu}\,d\tau\right)_{s=1}+c_{DB}S(1), \qquad (2)$$

where $c_{DB}S(1)$ is the base drag. The pressure p is given, to a consistent order of approximation, by

$$\frac{p-p_1}{\tfrac{1}{2}\rho_1 U^2}=-2\frac{\partial\phi_0}{\partial s}-\left(\frac{\partial\phi_0}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial\phi_0}{\partial\theta}\right)^2, \qquad (3)$$

where $z=re^{i\theta}$.

Strictly, this theory can only be applied to bodies of maximum thickness t, where t is small compared with unity, for which the angle which any tangent plane to the body boundary makes with the undisturbed stream direction is small and $O(t)$ and for which the rate of change of this angle is $O(t)$. Moreover, if d is the maximum diameter of a section of the body by a plane perpendicular to the stream, then the maximum curvature of this section must be $O(d^{-1})$ for all points where the section is convex outwards. This last condition is not met by the winged bodies considered in this paper, but it is shown in Ref. 1, p. 90, that the solution obtained

is an approximation to the true potential except in the neighbourhood of points where the curvature is infinite (*i.e.* the wing edges); furthermore, as the pressure singularities are integrable the expressions for the body forces should be true approximations for small incidences. It is difficult to say for what range of aspect ratio the theory is valid and it would seem that this can only be determined by means of an extended theory or experimental evidence.

Notation

U	velocity of the undisturbed stream
t	thickness ratio of the body
$U\phi$	perturbation potential
$U\phi_0$	approximate perturbation potential
ψ, ψ_0	complex conjugates of ϕ and ϕ_0
w_0	$\phi_0 + i\psi_0$
x, y, s	rectangular Cartesian co-ordinates
z	$x+iy$
a_0, b_0, a_N	functions of s only (N from 1 to ∞) in the expansion of w_0 (*see* Introduction)
$S(s)$	cross-section area of the body and wings by a plane $s=$constant
x_g, y_g	co-ordinates of the centre of area of this section
z_g	x_g+iy_g
Γ	contour of this section in the plane $s=1$
ν, τ	outward normal and tangent to Γ respectively
ρ_1	density of the fluid in the undisturbed stream
X, Y	components of total force in the directions of x and y increasing respectively
F	$X+iY$
D	drag force in the direction of s increasing
c_{pB}	coefficient of base drag
σ, λ	variables of integration
p	pressure
p_1	pressure in the undisturbed stream
r, θ	polar co-ordinates in x, y-plane
d	maximum diameter of body section by a plane perpendicular to the stream

$2f(y, s)$	thickness of the wing at (y, s)
$S_B(s)$	cross-section area of body alone
$c(s)$	body radius
$a(s)$	wing semi-span (Sections 2 and 3); height of fin (Section 4)
n	outward normal to body and wings
ϕ_1	$\phi_0 - (S_B'(s)\log r)/(2\pi)$
$F(y)$	strength of line source along $x=0$ } (*see* Section 2)
Φ	strength of image source } (*see* Section 2)
$\mathscr{R}, \mathscr{I}$	real and imaginary parts
M	Mach number of undisturbed flow
z_1	$z+\alpha s$
x_1, y_1	real and imaginary parts of z_1
α	incidence of body
β	incidence of wings (Section 3), of fin (Section 4)
γ	$\beta-\alpha$, incidence of wing to body
w_1	complex potential corresponding to ϕ_1
w_2	w_0-w_1
r_1, θ_1	polar co-ordinates in x_1, y_1-plane
ϕ_2	value of ϕ corresponding to w_2
ζ	z_1-c^2/z_1
ξ, η	real and imaginary parts of ζ
R'	$2c$
a'	$a+c^2/a$
u_2, v_2	$\partial\phi_2/\partial\xi$, $\partial\phi_2/\partial\eta$
q_2	u_2-iv_2
A', B', A, B, C	functions of R', a', α and β (*see* equations (18) and (19))
L	lift of system
Λ	lift when $\alpha=\beta$
L_x	extra lift due to incidence γ of wing (or fin) to body
L_B	lift of body alone
L_W	lift of wing alone
L_F	lift of fin alone
D_0	drag on system in position of zero lift

α'	incidence of system from zero lift position
D_B	drag on body alone
α_0	incidence of body in zero lift position
D_1	drag when $\beta=0$
b	maximum value of $a(s)$
R_1	value of $c(s)$ where maximum value of $a(s)$ occurs
K_1, K_2	functions of b and R_1, defined between equations (23) and (24)
B_1, C_1, d_1, d_2, d_3	functions of α, β, R' and a' defined by equations (28) and (29)
K	constant of integration in equation (30)
Ψ	$[(\zeta+iR')(\zeta-ia')]^{\frac{1}{2}}$
$S_w(s)$	area of wing section by plane $s=$constant
μ	*see* Fig. 2 and Appendix
k	$(1/\pi)(da/ds)\tan\mu$
a^*, y^*	a/c, y/c
b_{0B}, b_{0W}	contributions to b_0 from body and wing respectively
p_B	pressure due to body alone

2. A Body of Revolution with Wings of Symmetrical Section and of Small Aspect Ratio at Zero Incidence

The wings are assumed to be symmetrical about their chord plane $x=0$ and to be sufficiently thin to enable the boundary conditions on the wing surfaces to be applied on $x=0$ instead, *i.e.* their thickness must be small compared with the body radius. Let the radius of the body be $c(s)$, the semi-span of the wings $a(s)$, the thickness of the wing at the point (y, s) be $2f(y, s)$ and the area of the body section $S_B(s)$.

The appropriate two-dimensional problem is as follows. To find a solution ϕ_0 of $\nabla^2\phi_0=0$ in the plane $s=$constant such that

$$\frac{\partial\phi_0}{\partial n}=\frac{S'_B(s)}{2\pi c(s)} \quad \text{on} \quad |z|=c(s), \qquad (4)$$

$$\frac{\partial\phi_0}{\partial n}=\frac{\partial}{\partial s}[f(y,s)]=f'(y,s) \quad \text{say, on} \quad c<|y|<a, x=0, \qquad (5)$$

$$\partial\phi_0/\partial r\to 0 \quad \text{as} \quad r\to\infty, \qquad (6)$$

where n is the outward normal to the boundary of the section of the body and

wings by the plane $s=$constant. The dependence of a, c and $f'(y)$ upon s does not affect the solution of the two-dimensional problem and indication of the functional dependence will be omitted. The boundary conditions may be satisfied by placing a distribution of sources and sinks along the line $x=0$. To satisfy condition (4) a source of strength $S'_B(s)/(2\pi)$ is placed at the origin. If $\phi_1=\phi_0-(S'_B(s)\log r)/(2\pi)$, the function ϕ_1 satisfies the boundary conditions;

$$\partial\phi_1/\partial n=0 \text{ on } |z|=c, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\partial\phi_1/\partial n=f'(y) \text{ on } c<|y|<a,\ x=0, \quad . \quad . \quad . \quad (8)$$

$$\partial\phi_1/\partial r\to 0 \text{ as } r\to\infty. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

ϕ_1 is determined by placing a line source of strength $F(y)$ along the line $x=0$ from $y=-a$ to $y=-c$ and from $y=c$ to $y=a$. To satisfy condition (7) an image line source must be added from $y=-c$ to $y=-c^2/a$ and from $y=c^2/a$ to $y=c$ together with a sink of appropriate strength at the origin. The strength, Φ, of the image source is given by

$$F(y)\,dy=-\Phi(c^2/y)\,d(c^2/y),$$

i.e.

$$F(y)=(c^2/y^2)\,\Phi(c^2/y),$$

and the strength of the sink by

$$\int_c^a F(y)\,dy.$$

$F(y)$ is now determined in terms of $f'(y)$ by the use of equation (8). Let w_1 be the complex potential corresponding to ϕ_1; then

$$w_1=\int_c^a F(\lambda)\log(z^2+\lambda^2)\,d\lambda+\int_{c^2/a}^{c}F\left(\frac{c^2}{\lambda}\right)\frac{c^2}{\lambda^2}\log(z^2+\lambda^2)\,d\lambda-2\log z\int_c^a F(\lambda)\,d\lambda+b_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{dw_1}{dz}=2z\int_c^a\frac{F(\lambda)}{z^2+\lambda^2}\,d\lambda+2z\int_{c^2/a}^{c}\frac{c^2F(\lambda)}{(z^2+\lambda^2)\lambda^2}\,d\lambda-\frac{2}{z}\int_c^a F(\lambda)\,d\lambda. \quad . \quad . \quad . \quad . \quad (11)$$

Equation (8) gives

$$\mathscr{R}\left[\frac{dw_1}{dz}\right]_{z=iy+0}=f'(y) \text{ on } c<|y|<a,\ x=0+.$$

From (11), assuming that $c<|y|<a$ for definiteness

$$\left[\frac{dw_1}{dz}\right]_{z=iy+0} = 2ic^2 y \int_{c^2/a}^{c} \frac{F(\lambda)\,d\lambda}{\lambda^2(\lambda^2-y^2)} + \frac{2i}{y}\int_{c}^{a} F(\lambda)\,d\lambda + 2iy \int_{c}^{a} \frac{F(\lambda)\,d\lambda}{(\lambda^2-y^2)} + \pi F(y),$$

where the Cauchy Principle Value is to be taken in the third integral.

Hence,
$$f'(y) = \pi F(y). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Thus the complex potential w_0 is given by

$$w_0 = \frac{1}{\pi}\int_{c}^{a} f'(\lambda)\log(z^2+\lambda^2)\,d\lambda + \frac{c^2}{\pi}\int_{c^2/a}^{c} f'\left(\frac{c^2}{\lambda}\right)\log(z^2+\lambda^2)\frac{d\lambda}{\lambda^2} -$$

$$-\frac{2}{\pi}\log z \int_{c}^{a} f'(\lambda)\,d\lambda + \frac{1}{2\pi}S'(s)\log z + b_0. \quad . \quad (13)$$

The term b_0 is a constant in the two-dimensional problem, but is in general a function of s. It is shown in Ref. 1 that b_0 is given by

$$b_0 = \frac{1}{2\pi}\left\{S'(s)\log\{\tfrac{1}{2}(M^2-1)^{\frac{1}{2}}\} - \int_{0}^{s}\log(s-\sigma)S''(\sigma)\,d\sigma\right\}, \quad . \quad (14)$$

where M is the Mach number of the undisturbed flow.

Knowing w_0 and b_0 the pressure distribution on the body and wings can be calculated, and by a combination of this solution with that obtained in Section 3 for plane wings at small incidence the pressure distribution is obtained for a body of revolution with wings of symmetrical section at small incidence.

The pressure distribution on the wing has been computed for a body of revolution with wings of parabolic plan form and the results are given in the Appendix.

3. A Body of Revolution with Plane Wings of Small Aspect Ratio at Incidence, the Whole Being at Incidence to the Stream

Let α be the incidence of the body to the stream and β that of the wings; it is assumed that $|\beta-\alpha|$ is sufficiently small to permit the boundary conditions on the wing surfaces to be applied on the plane $x=-\alpha s$ instead. It will be necessary to assume, as in Ref. 1, that the body is of one of the following forms:

(*i*) a body of revolution of general variation of cross-section with wings of which the edges are everywhere leading edges, upstream from the base;

(*ii*) a body of revolution of uniform cross-section downstream from the section of maximum wing span.

These restrictions are necessary because downstream of the section of maximum span a vortex sheet is formed and the determination of the solution in the general case when the body radius is not constant there is very tedious.

Let $z_1=x_1+iy_1=z+\alpha s$ and let $\theta_1=\arg z_1$. The boundary conditions on w_0 in a plane $s=$constant which meets the leading edge of the wing are

$$\left.\begin{aligned} &\partial\phi_0/\partial n=-\alpha\cos\theta_1+S'(s)/(2\pi c) \text{ on } |z_1|=c,\\ &\partial\phi_0/\partial n=\mp\beta \text{ on } x_1=0\pm,\ c<|y_1|<a,\\ &|dw_0/dz_1|=O(1/|z_1|) \text{ at infinity}. \end{aligned}\right\} \quad . \quad . \quad (15)$$

The case $\alpha=\beta$ is solved in Ref. 1, the solution being

$$w_1-b_0=[1/(2\pi)]\,S'(s)\log z_1+\beta\{[(z_1+c^2/z_1)^2+(a-c^2/a)^2]^{\frac{1}{2}}-z_1\}.$$

If $w_0-w_1=w_2$, the boundary conditions on w_2 are

$$\left.\begin{aligned} &\partial\phi_2/\partial n=-(\alpha-\beta)\cos\theta_1 \text{ on } |z_1|=c,\\ &\partial\phi_2/\partial n=0 \text{ on } x_1=0,\ c<|y_1|<a,\\ &|dw_2/dz_1|=O(1/|z_1|^2) \text{ at infinity}. \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

The conformal transformation

$$\zeta=\xi+i\eta=z_1-c^2/z_1$$

is now applied.

The body section is mapped onto the imaginary axis between

$$\zeta=-(a+c^2/a)\,i \text{ and } \zeta=(a+c^2/a)\,i.$$

$$|d\zeta/dz_1|=2\,|\cos\theta_1| \text{ on } |z_1|=c.$$

The boundary conditions on w_2 in the ζ-plane are therefore

$$\left.\begin{aligned} &\partial\phi_2/\partial n=\pm(\beta-\alpha)/2 \text{ on } \xi=0\pm,\ |\eta|<2c,\\ &\partial\phi_2/\partial n=0 \text{ on } \xi=0,\ 2c<|\eta|<a+c^2/a,\\ &|dw_2/d\zeta|=O(1/|\zeta|^2) \text{ at infinity}. \end{aligned}\right\} \quad . \quad . \quad (17)$$

Let $R'=2c$, $a'=a+c^2/a$, $u_2=\partial\phi_2/\partial\xi$, $v_2=\partial\phi_2/\partial\eta$ and let $q_2=u_2-iv_2$ be the complex velocity. The boundary conditions (17) show that $\partial u_2/\partial\eta=0$ on $\xi=0$, $|\eta|<a'$ and therefore $dq_2/d\zeta$ must be real on this segment. q_2 would be expected to have a branch point at the tips of the wing and $dq_2/d\zeta$ to have poles at the points where the wings meet the body. Hence, remembering that $|dq_2/d\zeta|=O(|\zeta^{-2}|)$,

$$\frac{dq_2}{d\zeta}=\frac{A'\zeta^2+B'}{(\zeta^2+R'^2)(\zeta^2+a'^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (18)$$

where A' and B' are real constants; it can be seen from symmetry that there is no term in ζ in the numerator.

Integrating, we obtain

$$q_2=\frac{A\zeta}{(\zeta^2+a'^2)^{\frac{1}{2}}}+B\cosh^{-1}\left[\frac{R'}{a'}\left(\frac{\zeta^2+a'^2}{\zeta^2+R'^2}\right)^{\frac{1}{2}}\right]+C. \quad . \quad . \quad (19)$$

A is real and B is imaginary. A, B and C are now determined from the boundary conditions. The jump in the imaginary part of the term $\cosh^{-1}[R'(\zeta^2+a'^2)^{\frac{1}{2}}/\{a'(\zeta^2+R'^2)^{\frac{1}{2}}\}]$ on the upper surface of the wing at $\zeta=iR'$ is $\frac{1}{2}\pi i$. Thus, since the jump in $\mathscr{R}[q_2]$ is $\frac{1}{2}(\alpha-\beta)$,

$$B=i(\beta-\alpha)/\pi,$$

and, since $\mathscr{R}[q_2]=\frac{1}{2}(\beta-\alpha)$ at $\zeta=0$,

$$\mathscr{R}[C]=\tfrac{1}{2}(\beta-\alpha).$$

The condition $q_2=0$ at infinity yields

$$A+B\cosh^{-1}(R'/a')+C=0,$$

that is $A=(\alpha-\beta)\{\frac{1}{2}-(1/\pi)\cos^{-1}(R'/a')\}$ and $\mathscr{I}[C]=0$.

It follows that q_2 is given by

$$q_2=(\beta-\alpha)\left\{\left(\frac{1}{\pi}\cos^{-1}(R'/a')-\tfrac{1}{2}\right)\frac{\zeta}{(\zeta^2+a'^2)^{\frac{1}{2}}}+\frac{i}{\pi}\cosh^{-1}\left[\frac{R'}{a'}\left(\frac{\zeta^2+a'^2}{\zeta^2+R'^2}\right)^{\frac{1}{2}}\right]+\tfrac{1}{2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

To find ϕ_2 further integration is necessary, but the lift can be obtained without integration since only a_1, the coefficient of $1/z$ in the expansion of w_0 need be known. The coefficient of ζ^{-2} in the expansion of q_2 in descending powers of ζ^2 is

$$-\tfrac{1}{2}a'^2A+\tfrac{1}{2}iR'B(a'^2-R'^2)^{\frac{1}{2}},$$

which is equal to

$$\{ (\beta-\alpha)/(2\pi) \}\{ a'^2 \sin^{-1}(R'/a') - R'(a'^2-R'^2)^{\frac{1}{2}} \} .$$

This coefficient is equal to the coefficient of $-1/z$ in the expansion of w_2. The coefficient of $1/z$ in the expansion of w_1 is

$$\tfrac{1}{2}\alpha s S'(s)/\pi + \tfrac{1}{2}\beta(a^4+c^4)/a^2,$$

and hence

$$a_1 = \frac{\alpha s S'(s)}{2\pi} + \frac{\beta}{2}\left[\frac{a^4+c^4}{a^2}\right] - \frac{(\beta-\alpha)}{2\pi}\left[\frac{(a^2+c^2)^2}{a^2}\sin^{-1}\left(\frac{2ca}{a^2+c^2}\right) - \frac{2c(a^2-c^2)}{a}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

As mentioned earlier, this expression holds only so long as the wing edges are leading edges; however, it is shown in Ref. 1 that the solution is of the same form downstream of the section of maximum span, but a, the semi-span, must be replaced by b, the maximum semi-span and the body radius c by its value, R_1, at the section of maximum span. The same is true in this case; the strength of the vortex sheet is independent of s and hence of a. Thus a solution is obtained which is identical with that for a wing which has constant span downstream of the section of maximum span. It should be noted that this is only true so long as one of the conditions (*i*) and (*ii*) given at the beginning of this Section is met.

The lift on the system is given by

$$\frac{L}{\frac{1}{2}\rho_1 U^2} = 4\pi(a_1)_{s=1} + 2S'(1)z_s(1) + 2S(1)z_s'(1)$$

$$= 2\pi\beta\left[\frac{b^4+R_1^4}{b^2}\right] - 2\pi R_1^2\alpha + 2(\alpha-\beta)\left[\frac{(b^2+R_1^2)^2}{b^2}\sin^{-1}\left(\frac{2bR_1}{b^2+R_1^2}\right) - \frac{2R_1}{b}(b^2-R_1^2)\right] . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This may be written

$$\frac{L}{\frac{1}{2}\rho_1 U^2} = \frac{\Lambda}{\frac{1}{2}\rho_1 U^2} + 2(\alpha-\beta)\left[\frac{(b^2+R_1^2)^2}{b^2}\sin^{-1}\left(\frac{2bR_1}{b^2+R_1^2}\right) - \frac{2R_1}{b}(b^2-R_1^2) - \pi\frac{b^4+R_1^4}{b^2}\right], \quad (23)$$

where Λ is the lift when both body and wing are at incidence α. Thus the extra

lift, L_x, due to the incidence γ of the wing to the body is given by

$$\frac{L_x}{\frac{1}{2}\rho_1 U^2} = 2\gamma\left[\pi\frac{b^4+R_1^4}{b^2} + \frac{2R_1}{b}(b^2-R_1^2) - \frac{(b^2+R_1^2)^2}{b^2}\sin^{-1}\left(\frac{2bR_1}{b^2+R_1^2}\right)\right].$$

The effective lift of the wings is always increased by the presence of the body, so long as the incidence of the wing is greater than that of the body. If L, L_B, and L_W denote the lifts of the system, body, and wing respectively, we have[1]

$$L_B/(\tfrac{1}{2}\rho_1 U^2) = 2\pi R_1^2\,\alpha,\quad L_W/(\tfrac{1}{2}\rho_1 U^2) = 2\pi(b-R_1)^2\,\beta.$$

From (22), it can be seen that

$$\frac{L-L_B}{\frac{1}{2}\rho_1 U^2} = 2\pi\beta\left[\frac{b^4-2R_1^2b^2+R_1^4}{b^2}\right] + 2(\alpha-\beta)\left[\frac{(b^2+R_1^2)^2}{b^2}\sin^{-1}\left(\frac{2bR_1}{b^2+R_1^2}\right) - 2\pi R_1^2 - \frac{2R_1(b^2-R_1^2)}{b}\right].$$

Hence, using the fact that $\alpha<\beta$ and the inequality $\sin^{-1}x \leqslant \frac{1}{2}\pi x$, we have

$$L - L_B \geqslant L_W(1+R_1/b+R_1^2/b^2).$$

The drag may be calculated from the formula (Ref. 1, p. 90)

$$D = D_0 + \tfrac{1}{2}\alpha' L,$$

where D_0 is the drag in the position of zero lift and α' is the incidence of the system from that position. Writing equation (22) as

$$L/(\tfrac{1}{2}\rho_1 U^2) = K_1\alpha + K_2\gamma,$$

where $$K_1 = 2\pi(b^4 - b^2R_1^2 + R_1^4)/b^2$$

and $$K_2 = 2\left[\pi\frac{b^4+R_1^4}{b^2} - \frac{(b^2+R_1^2)^2}{b^2}\sin^{-1}\left(\frac{2bR_1}{b^2+R_1^2}\right) + \frac{2R_1}{b}(b^2-R_1^2)\right],$$

we have, if α_0 is the incidence of the body in the position of zero lift

$$K_1\alpha_0 + K_2\gamma = 0.$$

Hence $$L/(\tfrac{1}{2}\rho_1 U^2) = K_1(\alpha-\alpha_0) = K_1\alpha',$$

and thus $$D/(\tfrac{1}{2}\rho_1 U^2) = D_0/(\tfrac{1}{2}\rho_1 U^2) + \tfrac{1}{2}K_1\alpha'^2. \quad . \quad . \quad . \quad (24)$$

To determine D_0 it is most convenient to determine first D_1, the drag when $\beta=0$. From equation (2) it may be seen that D_1 is the sum of two terms, the drag of the body alone at zero incidence and that part of the integral $-\int_\Gamma \phi_0(\partial\phi_0/\partial\nu)d\tau$ which depends on α. In general the wing (or vortex sheet) must be included in the contour Γ; however, in this case the integral over the wing is zero, since on it $\partial\phi_0/\partial\nu=\pm\beta=0$. The contour Γ may be taken to be the circle $z_1=R_1e^{i\theta_1}$; on this circle

$$d\tau=R_1\,d\theta_1=-iR_1\,dz_1/z_1$$

and

$$\partial\phi_0/\partial\nu=-\alpha\cos\theta_1+S'(1)/(2\pi R_1)$$

$$=-\alpha(z_1^2+R_1^2)/(4z_1R_1)+S'(1)/(2\pi R_1).$$

Thus

$$\int_\Gamma \phi_0\frac{\partial\phi_0}{\partial\nu}d\tau=\mathscr{R}\left[\int_\Gamma iw_0\left[\frac{\alpha}{2}\left(\frac{z_1^2+R_1^2}{z_1}\right)-\frac{S'(1)}{2\pi}\right]\frac{dz_1}{z_1}\right].$$

The term dependent on α is found to be

$$-\pi\alpha(a_1)_{\beta=0}+\tfrac{1}{2}\alpha^2S'(1).$$

Hence,

$$D_1/(\tfrac{1}{2}\rho_1U^2)=\pi\alpha(a_1)_{\beta=0}-\tfrac{1}{2}\alpha^2S'(1)+D_B/(\tfrac{1}{2}\rho_1U^2),$$

where D_B is the drag on the body alone at zero incidence. Using equation (24),

$$\frac{D_0}{\frac{1}{2}\rho_1U^2}=\frac{D_B}{\frac{1}{2}\rho_1U^2}+\frac{\gamma^2}{2}\left[\frac{(b^2+R_1^2)^2}{b^2}\sin^{-1}\left(\frac{2bR_1}{b^2+R_1^2}\right)-\frac{2R_1(b^2-R_1^2)}{b}-\frac{(K_2-K_1)^2}{K_1}\right]$$

$$=\frac{D_B}{\frac{1}{2}\rho_1U^2}+\frac{\gamma^2}{4}\left[2\pi R_1^2-\frac{(2K_2-K_1)(K_2-K_1)}{K_1}\right]. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Therefore for any given body and given wing, if the lift is prescribed, the drag is a minimum when the body and wing are at the same incidence to the stream. To see this it may be noted that $\alpha'=L/K_1$, and hence for given lift α' is constant and from equation (24) the drag has a minimum when D_0 has a minimum, that is, when $\gamma=0$.

4. A Body of Revolution with a Plane Fin (i.e. a Wing on One Side Only) of Small Aspect Ratio Set at Incidence, the Whole Being at Incidence to the Stream

The method of solution is identical with that of Section 3. Let the incidence of the fin to the undisturbed stream be β and that of the body α; let the body radius

be $c(s)$ and the height of the fin $a(s)$. Let the fin lie in the plane $x=-\alpha s$ and let $z_1=x_1+iy_1=z+\alpha s$. The boundary conditions in the appropriate two-dimensional problem are

$$\left.\begin{aligned} &\partial\phi_0/\partial n=-\alpha\cos\theta_1+S'(s)/(2\pi c) \text{ on } z_1=ce^{i\theta_1},\\ &\partial\phi_0/\partial n=\mp\beta \text{ on } x_1=0\pm,\ c<y<a,\\ &|dw_0/dz_1|=0(|z_1|^{-1}) \text{ at infinity}. \end{aligned}\right\} \quad . \quad . \quad (26)$$

The case $\beta=\alpha$ is first solved. Let the solution to this problem be $w_1-\alpha z_1+S'(s)\log z_1/(2\pi)$; w_1 satisfies the boundary conditions

$$dw_1/dz_1=\alpha \text{ at infinity},$$

$$\partial\phi_1/\partial n=0 \text{ on } z_1=ce^{i\theta_1}, \text{ and on } x_1=0,\ c\leqslant y_1\leqslant a.$$

If the transformation $\zeta=z_1-c^2/z_1$ is now applied the problem is reduced to that of the flow of a uniform stream of velocity α past a flat plate at right angles to the stream; the length of the plate is $a(1+c/a)^2$ and its mid-point is at $\zeta=\frac{1}{2}ia(1-c/a)^2$.

Thus $$w_1=\alpha[\{\zeta-\tfrac{1}{2}ia(1-c/a)^2\}^2+\tfrac{1}{4}a^2(1+c/a)^4]^{\frac{1}{2}}.$$

As in Section 3 this holds only so long as the edge of the fin is a leading edge. To find the solution downstream of the section at which the maximum height of the fin occurs the same assumptions must be made regarding the body section as in Section 3; the solution is then found by replacing a by its maximum value b, and c by its value R_1, at the section at which a attains its maximum.

The coefficient of $1/z$ in the expansion of w_1 when $s=1$ is

$$\alpha\left[\frac{b^2}{8}\left(1+\frac{R_1}{b}\right)^4-R_1^2\right].$$

The coefficient of $1/z$ in the expansion of the full potential in descending powers of z is

$$\alpha\left[\frac{b^2}{8}\left(1+\frac{R_1}{b}\right)^4-R_1^2+\frac{S'(1)}{2\pi}\right], \quad . \quad . \quad . \quad (27)$$

and hence the lift is given by

$$L/(\tfrac{1}{2}\rho_1 U^2)=4\pi\alpha\{\tfrac{1}{8}b^2(1+R_1/b)^4-\tfrac{3}{2}R_1^2\}.$$

The lift of the body alone is given by

$$L_B/(\tfrac{1}{2}\rho_1 U^2)=2\pi\alpha R_1^2,$$

and that of the fin alone by

$$L_F/(\tfrac{1}{2}\rho_1 U^2)=\tfrac{1}{2}\pi\alpha(b-R_1)^2.$$

Thus
$$L=L_B+L_F(1+6R_1/b+R_1^2/b^2).$$

Returning to the case when $\beta\neq\alpha$, it is sufficient to solve the problem

$$\partial\phi_2/\partial n=(\beta-\alpha)\cos\theta_1 \text{ on } |z_1|=c,$$

$$\partial\phi_2/\partial n=0 \qquad \text{on } x_1=0,\ c<y<a,$$

$$|dw_2/dz_1|=O(|z_1|^{-2}) \qquad \text{at infinity},$$

for by adding the solution of this problem to the one obtained for the case $\beta=\alpha$ the general solution is obtained.

Again applying the conformal transformation

$$\zeta=\xi+i\eta=z_1+c^2/z_1,$$

the boundary conditions become

$$\partial\phi_2/\partial n=\pm\tfrac{1}{2}(\beta-\alpha) \text{ on } \xi=0\pm,\ -R'<\eta<R',$$

$$\partial\phi_2/\partial n=0 \qquad \text{on } \xi=0,\ R'<\eta<a',$$

$$|dw_2/d\zeta|=O(|\zeta|^{-2}) \quad \text{at infinity},$$

Let q_2 be the complex velocity $dw_2/d\zeta$. q_2 is expected to have branch points at $\zeta=-iR'$ and $\zeta=ia'$ and $dq_2/d\zeta$ to have a pole at $\zeta=iR'$. Hence we write

$$\frac{dq_2}{d\zeta}=\frac{B_1\zeta+C_1 i}{(\zeta-iR')(\zeta+iR')^2(\zeta-ia')^2} \quad . \quad . \quad . \quad . \quad (28)$$

On $\xi=0$, $-R'<\eta<a'$, $\partial q_2/\partial\eta$ is imaginary and hence B_1 and C_1 are real.

Let
$$\frac{B_1\zeta+C_1 i}{(\zeta-iR')(\zeta+iR')(\zeta-ia')}=\frac{d_1}{(\zeta-iR')}+\frac{d_2}{(\zeta+iR')}+\frac{d_3}{(\zeta-ia')} \quad . \quad . \quad (29)$$

d_1, d_2 and d_3 are determined from the boundary conditions and from the equation

$$d_1+d_2+d_3=0.$$

Integrating $dq_2/d\zeta$ it is found that

$$q_2 = \frac{d_1}{[2R'(a'-R')]^{\frac{1}{2}}} \log \frac{3a'R' - R'^2 + i\zeta(3R'-a') - 2\Psi[2R'(a'-R')]^{\frac{1}{2}}}{(\zeta - iR')} -$$

$$- \frac{2d_2 i\Psi}{(a'+R')(\zeta+iR')} + \frac{2d_3 i\Psi}{(a'+R')(\zeta - ia')} + K, \qquad (30)$$

where K is a constant of integration and $\Psi = [(\zeta+iR')(\zeta-ia')]^{\frac{1}{2}}$. The jump in the real part of q_2 at $\zeta = iR'$ is $\frac{1}{2}(\alpha-\beta)$ and this gives

$$d_1 = [(\alpha-\beta)/(2\pi)][2R'(a'-R')]^{\frac{1}{2}} i.$$

At $\zeta = 0$, $\mathscr{R}[q_2] = \frac{1}{2}(\beta-\alpha)$ and hence

$$\mathscr{R}\left[\frac{\alpha-\beta}{2\pi} i \log \frac{3a'R' - R'^2 - 2R'[2a'(a'-R')]^{\frac{1}{2}}}{-iR'}\right] + \mathscr{R}[K] = \tfrac{1}{2}(\beta-\alpha),$$

i.e.

$$\mathscr{R}[K] = \tfrac{1}{2}(\beta-\alpha).$$

The boundary condition at infinity yields

$$\frac{\alpha-\beta}{2\pi} i \log[(3R'-a')i - 2[2R'(a'-R')]^{\frac{1}{2}}] - \frac{2d_2 i}{(a'+R')} + \frac{2d_3 i}{(a'+R')} + K = 0.$$

Taking real and imaginary parts

$$d_3 - d_2 = \frac{\beta-\alpha}{4\pi}(a'+R')i\left\{\arg\left[(3R'-a')i - 2[2R'(a'-R')]^{\frac{1}{2}}\right] + \pi/2\right\},$$

and

$$\mathscr{I}[K] = \frac{\beta-\alpha}{2\pi}\log(a'+R').$$

The coefficient of $1/z$ in the expansion of w_2 is

$$\frac{i}{4}\left[(3R'-a')d_1 + \frac{\beta-\alpha}{4\pi}(a'+R')^2 i\left\{\arg\left[(3R'-a')i - 2[2R'(a'-R')]^{\frac{1}{2}}\right] + \pi/2\right\}\right].$$

Thus the extra lift due to the extra incidence, $\beta-\alpha$, of the fin to the body is

$$L_w = 4\pi(\beta-\alpha)\left\{\left(\frac{b^2}{8}\left[1+\frac{R_1}{b}\right]^4 - R_1^2\right) - \frac{(b^2 - 6bR_1 + R_1^2)}{4\pi b}\left[\frac{R_1}{b}(b-R_1)^2\right]^{\frac{1}{2}} -\right.$$

$$\left. - \frac{(b+R_1)^4}{16\pi b^2}\left[\arg\left(\left[\frac{6bR_1 - b^2 - R_1^2}{b} i\right] - 4\left[\frac{R_1}{b}(b-R_1)^2\right]^{\frac{1}{2}}\right) + \pi/2\right]\right\}.$$

ACKNOWLEDGMENT

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APPENDIX

The pressure distribution on the wings of a system consisting of a body of revolution with wings of symmetrical section, at zero incidence, has been calculated in two special cases.

The dimensions of the body are taken as follows. The cross-sectional area of the body is defined by

$$S_B(s) = (0.05)^2\pi + (0.08)\pi(s-0.5)^3[1+1.5(s-0.5)], \text{ if } 0 \leqslant s \leqslant 0.5$$

$$= (0.05)^2\pi, \text{ if } 0.5 \leqslant s \leqslant 1.$$

The body is chosen so that $S_B(s)$, $S'_B(s)$ and $S''_B(s)$ are continuous at $s=0.5$. The wing span, $a(s)$, is defined by

$$a-c=0, \text{ if } s \leqslant 0.5$$

$$= 1.6(s-0.5)(1-s), \text{ if } 0.5 \leqslant s \leqslant 1.$$

The section of a wing by a plane $s=$constant is taken to be a triangle, with angle 2μ at the tip, and μ is taken as $\tan^{-1}(0.1)$. The plan form is shown in Fig. 1 and a section in Fig. 2.

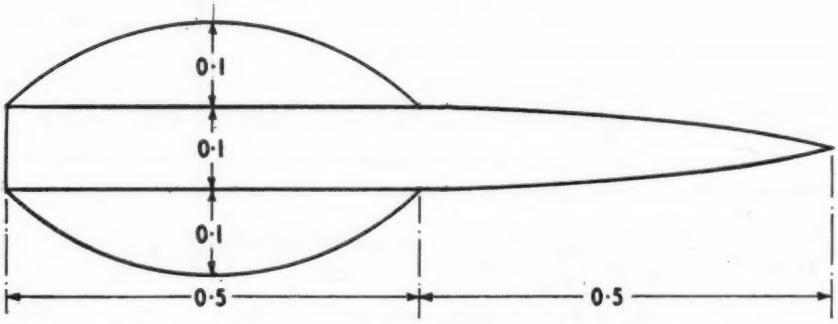

Fig. 1.

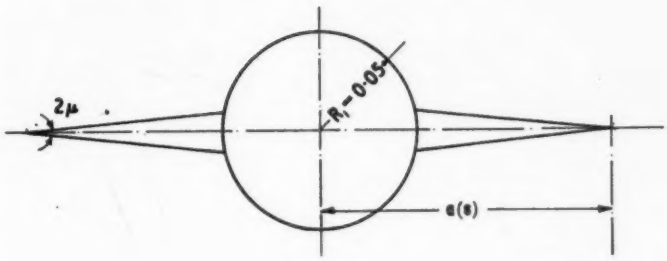

Fig. 2.

From equation (13) we find that

$$\frac{\partial \phi_0}{\partial s} - \frac{\partial b_0}{\partial s} = c\frac{dk}{ds}\left[a^* \log (a^{*2} - y^{*2}) - 2\log (y^{*2} - 1) - y^* \log\left(\frac{a^* - y^*}{a^* + y^*}\right) + y^* \log\left(\frac{y^* - 1}{y^* + 1}\right) - \right.$$

$$- 2(a^* - 1) - 2(a^* - 1)\log y + \frac{1}{y^*}\log\left(\frac{y^* - 1}{y^* + 1}\right) + a^* \log (a^{*2} y^{*2} - 1) -$$

$$\left. - \frac{1}{y^*}\log\left(\frac{a^* y^* - 1}{a^* y^* + 1}\right) - 2a^* \log a^* + 2(a^* - 1)\log c\right] + k\frac{da}{ds}[\log (a^{*2} - y^{*2}) +$$

$$+ \log (a^{*2} y^{*2} - 1) - 2\log y^* - 2\log a^* + 2\log c],$$

where $k = (1/\pi)(da/ds)\tan\mu$, $a^* = a/c$ and $y^* = y/c$.

$$\frac{\partial \phi_0}{\partial y} = k\left[\log\left(\frac{a^* - y^*}{a^* + y^*}\right) - \log\left(\frac{y^* - 1}{y^* + 1}\right) + \frac{1}{y^{*2}}\log\left(\frac{y^* - 1}{y^* + 1}\right) - \frac{1}{y^{*2}}\log\frac{a^* y^* - 1}{a^* y^* + 1}\right],$$

and $\partial \phi_0 / \partial x = \pi k$. b_0 is given by formula (14) and denoting by b_{0W} and b_{0B} the contributions to b_0 from the wings and body respectively, we find that

$$b'_{0B} = \frac{1}{2\pi}\left\{S''_B(s)\log[\tfrac{1}{2}(M^2 - 1)^{\frac{1}{2}}] - \int_0^s \log(s - \sigma) S'''_B(\sigma)\, d\sigma - S''_B(0)\log s\right\},$$

and

$$b'_{0W} = \frac{1}{2\pi}\left\{S''_W(s)\log[\tfrac{1}{2}(M^2 - 1)^{\frac{1}{2}}] - \int_{0.5}^s \log(s - \sigma) S'''_W(\sigma)\, d\sigma - S''_W(0.5)\log s\right\}.$$

In the computation the Mach number of the undisturbed stream is taken to be $\sqrt{5}$ and hence $(M^2 - 1)^{\frac{1}{2}} = 2$.

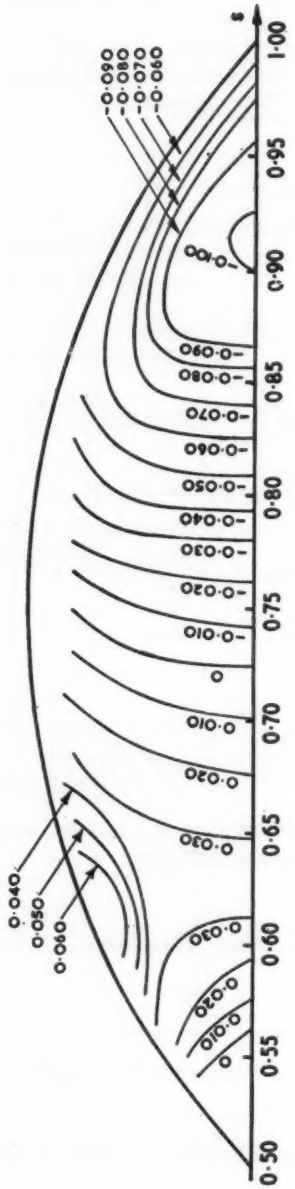
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Fig. 3.

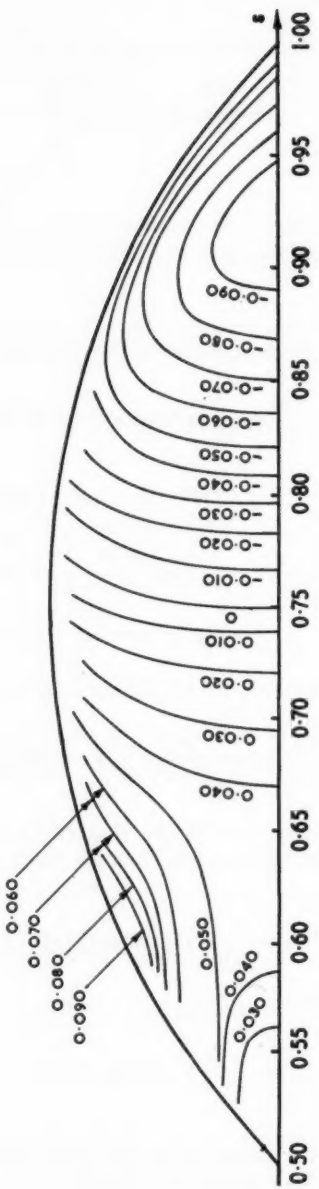
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Fig. 4.

The results are shown in Figs. 3 and 4. Lines of constant pressure on the wing are shown in Fig. 3, the values of $(p-p_0)/(\frac{1}{2}\rho_1 U^2)$ being indicated. In Fig. 4 lines upon which $(p-p_B)/(\frac{1}{2}\rho_1 U^2)$ is constant are shown, p_B being the pressure due to the body alone. The solution gives an infinite pressure at the wing edge, and the isobars have not been computed in the neighbourhood of the edge.

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Normal Elliptic Functions: A Normalized Form of Weierstrass's Elliptic Functions *by A. R. Low, M.A. University of Toronto Press. Toronto,* 1950. *Price* $1.25 *(Can.).*

The general quartic, with four complex roots, may be transformed by homographic transformation to a cubic with three complex roots and one root at infinity. The further arbitrary condition that the sum of the roots is zero gives Weierstrass's standard form. When the roots of the cubic are real the form is that which arises naturally in a class of physical problems referred to a system of confocal real conicoids, ellipses, one-sheet hyperboloids and two-sheet hyperboloids.

The quartic was normalised by Legendre to the form

$$S=(1-s^2)(1-k^2 s^2) \qquad 0<k^2<1 .$$

with only one real arbitrary parameter k.

In analogous manner the cubic is normalised by the author to the form

$$P=p(p-m)(p-1) \qquad 0<m<1 .$$

Taking $m=k^2$, the substitution $p=s^{-2}$ transforms Weierstrass' elliptic integral with normalised cubic to Legendre's first elliptic integral

$$u=\tfrac{1}{2}\int_{p}^{\infty} P^{-\frac{1}{2}}\,dp=\int_{0}^{s} S^{-\frac{1}{2}}\,ds .$$

The real p-axis is divided into four segments by the roots of the normal cubic, 0, m, 1, and the concealed root at infinity, and four elliptic integrals are defined, one in each range, denoted by suffixes, p_1, p_2, p_3, p_4, being regarded as independent variables. Inverting the functional relations between u and p, four elliptic functions are defined and may be written $p_1\,(u,m)$, $p_2\,(u,m)$, $p_3\,(u,m)$, $p_4\,(u,m)$, to denote the functional nature of p explicitly, u being now regarded as the independent variable.

The equation of the confocal system becomes

$$x^2 p^{-1}+y^2(p-m)^{-1}+z^2(p-1)^{-1}=1 .$$

Putting $p=1$, a limiting form is obtained as an elliptic lamina

$$x^2+y^2(1-m)^{-1}=1$$

which may be taken as the conicoid of reference, with unit semi-diameter and only one arbitrary parameter, m=eccentricity squared.

This is analogous to referring the elementary transcendental functions sin u, cos u, sinh u, cosh u, to a circle and a rectangular hyperbola with unit semi-diameters, with no arbitrary parameter.

(It is remarkable that these two analogies have escaped the attention of mathematicians, possibly from their preoccupation with the cult of general mathematical dialectics, during the last three quarters of a century.)

Simple relations are established

$$p_1 p_3 = m\,,\quad p_3 + p_{2'} = 1\,,\quad p_1 - p_{4'} = 1$$

accents on the suffixes 2′, 4′, denoting that the complementary parameter $m' = 1 - m$ replaces m in the functions thus denoted.

The substitution $p = s^{-2}$ gives corresponding textbook relations between s_1, s_2, s_3, s_4, expressed in terms of sn u, cn u, dn u, including Jacobi's imaginary transformation.

The term "modulus" is used indiscriminately in textbooks for the real positive quantity of the same numerical magnitude as a complex quantity under consideration, for the parameters m and k, and for the periods ω, ω'.

In the paper its use is restricted to the first definition, and all quantities with implicit complex signs are replaced by their modulus affected by a wholly explicit complex sign of the form $e^{i\theta}$, called the generalised sign, θ being called the phase of the complex number. The changes of the generalised sign as the path of integration passes the singular points of $P^{-\frac{1}{2}}$ at the roots ∞, 1, m, 0, are established in the smallest detail and the results are tabulated against the quantities which they affect. In this way the discussion of each case as it arises, often difficult, is avoided.

Every quantity discussed is shown graphically and extensive use is made of the Argand diagram.

The paper thus far is commended to the attention of students of physics and engineering as a logical introduction to the simpler forms of elliptic functions which are immediately amenable to numerical applications. The tables of the p-functions at the end of the paper avoid the hitherto necessary labour of converting to Legendre's elliptic integrals, or Jacobi's sn u, for which tables are available.

In Sections 23 to 25 examples are discussed which call for a more extended knowledge of the subject.

In Sections 26 and 27 general methods are used to establish the additional theorem and its application to any point in the whole complex plane, with a method

of expressing the result in terms of the p-functions and thus making numerical applications, by use of the tables at the end of the paper.

In the last Section it is stated that the zeta and sigma functions may be normalised in the same way, and that Lamé's polynomials in their normalised form may be computed and tabulated without prohibitive labour.

This opens a new field of computation and raises the question of demand and supply. In the case of Bessel functions, computation of the simpler forms has been followed by more and more generalised forms and now absorbs a predominant share of computing activities. It may also be recalled that problems of wind tunnel interference have been solved in terms of theta functions, supply in this case anticipating demand by more than a century. The question will no doubt receive due consideration by computing bodies and institutions.